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MISSING PREIMAGES FOR CHAOTIC LOGISTIC MAP WITH A HOLE

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Chaotic transients and preimages are investigated for a new map proposed recently, having a hole in the unit interval of the $r = 4$ logistic map. This map is characterized by deviations from Frobenius-Perron equation for average lifetimes in dependence on hole position in the form of bursts of average lifetime. We present classification of these bursts on the basis of average lifetimes. Using time maps it is investigated how these bursts are caused by missing preimages of the hole interval $I^{(0)}$. We derive approximate expression for the ratio of lifetimes deduced from Frobenius-Perron and from Kantz-Grassberger equations.

1. Introduction

Transient chaos plays an important role in various dynamical systems [1]. It was observed in the Lorenz model [2–5], low-dimensional maps [6–9], nonlinear oscillators [10,11], delay equations [12], partial differential equations [13,14], coupled map lattices [15,16], one-dimensional robot model [17,18], discrete gene sequences in biology [19], models of amorphous solids [20] and of quasicrystals [21], neurobiology [22], epidemiology [23], laser with modulated losses [24] etc. It was found that the phenomemon of chaotic chaos is associated with dynamical system passing through crisis [25]. It is observed in a dynamical system in the range of control parameter λ which is slightly above the critical value λ_c where the crisis takes place. For $\lambda > \lambda_c$, the character of time evolution of the system is connected with the coexistence of at least one strange repeller and one attractor. Trajectories initiated in the neighborhood of a repeller are bouncing chaotically between the points of a repeller during a finite time after which they leave this region, then approach the attractor and stay on it. Thus, the asymptotic behavoir of a system is determined by the properties of final attractor. In a case of many repellers coexisting with one final attractor one observes a cascade [26] of chaotic transients ending on the attractor.

2. *Chaotic transienst for the $r = 4$ logistic map with a narrow hole*

The significance of one-dimensional maps is underscored by the fact that new ideas about general chaotic systems are often investigated and exemplified on one-dimensional maps. A further motivation for studying one dimensional maps is that higher-dimensional systems often approximately reduce to one-dimensional dynamics. Thus, the one-dimensional maps are well suited for studying transient chaos not only since they provide the simplest examples but also because they model very closely what is going on along the unstable manifolds of strange chaotic sets in higher-dimensional systems with an one-dimensional unstable manyfold. It is possible to model transient chaos by using some special maps.

Transient chaos occurs for one-dimensional maps if a unit interval I is mapped under the dynamics $f(x)$ not only onto itself but but partially also outside itself [1]. An example of such a map is the logistic map [1,27] in the case of control parameter r being larger than 4.

Recently we have introduced [28,29] a new map producing chaotic transients defined on the unit interval by

$$f(x) = \begin{cases} 4 \cdot x \cdot (1-x), & x \notin I^{(0)} \\ \infty, & x \in I^{(0)} \end{cases}, \tag{1}$$

where $I^{(0)}$ is a narrow interval $(\xi - \frac{d}{2}, \xi + \frac{d}{2})$ within the unit interval I.

For $r = 4$ the logistic map maps the unit interval I onto itself, which is the case of fully developed chaos, expessed by an exact solution [30]

$$x_n = \sin^2(2^n \arcsin\sqrt{x_0}) \tag{2}$$

where $0 \leq \arcsin\sqrt{x_0} \leq \pi$ and the invariant density is

$$W(x) = \frac{1}{\pi\sqrt{x \cdot (1-x)}}. \tag{3}$$

Our map (1) coincides with the $r = 4$ logistic map on the unit interval, except for a narrow hole of the width d at the position $x = \xi$. In the first iteration the interval $I^{(0)}$ around ξ escapes the unit interval I, in the second iteration its first preimages escape, and so on (Fig. 1). Finally, we end up with an invariant strange set in I, which is a chaotic repeller. On the other hand, any trajectory escaping the unit interval I goes to the attractor at infinity.

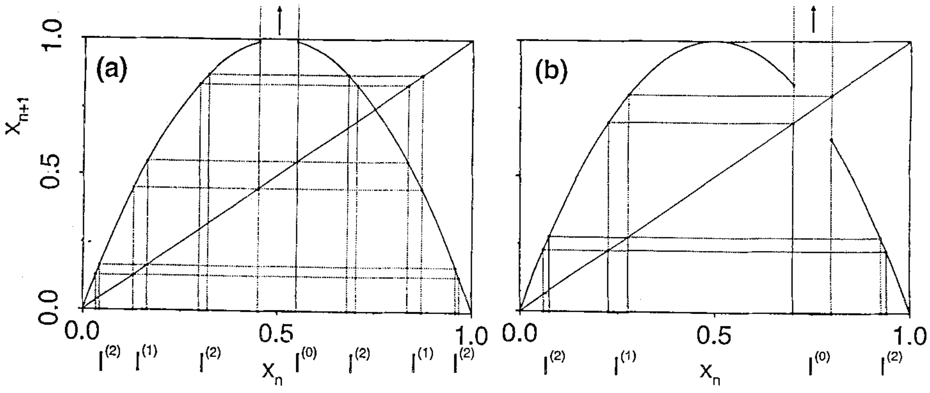

Fig. 1. The new map (1) and the $I^{(1)}$, $I^{(2)}$ preimages of $I^{(0)}$. The hole is defined by (a) $\xi = 0.5$, $d = 0.1$ and (b) $\xi = 0.75$, $d = 0.1$.

In Figs. 2.a-e we present, as illustrations, trajectories corresponding to chaotic transients. In figure the attractor in which after initial transients the trajectory ends up is not at ∞, as for the map (1), but is determined for the case of a map with the constant value $f(x) = const.$ within the hole, where constant value is taken as $\frac{1}{2}[f(\xi - \frac{d}{2}) + f(\xi + \frac{d}{2})]$. In this case instead of attractor of period 1 at infinity, we have attractors of various periods. In the cases considered we have periods 1, 2, 3, 4 and 52, respectively. Of course, the trajectory of a transient does not depend on the definition of $f(x)$ within the hole.

Illustrative examples of distributions of lifetimes of chaotic transients are displayed in Fig. 3, which presents distribution. The number of surviring orbits as a function of time (i.e. iteration steps) is illustrated in Fig. 3. From the ensuing exponential distribution we determine the average lifetime.

In analogy to Ref. 28 we have calculated the averege lifetime of chaotic transient associated with the map (1) in dependence on the hole position ξ for the hole width $d = 0.03$ (Fig. 4). The average lifetime is calculated for each of the positions of the hole interval $I^{(0)}$: $(0,d)$, $(\frac{1}{10}d, \frac{11}{10}d)$, $(\frac{2}{10}d, \frac{12}{10}d)$, ...,$(1-d,1)$. In each case the average lifetime τ was determined from an exponential decay of the number of survivors

$$N_n = N_0 e^{-\kappa n} \tag{4}$$

where N_n denotes the number of orbits staying still inside the unit interval I after n steps, N_0 is a constant corresponding to the initial number of trajectories (we take $N_0 = 10^6$ i.e. 10^6 inital positions uniformly distributed along the unit interval) and κ is escape rate. (More precisely, as noted previously, for small intervals N_0

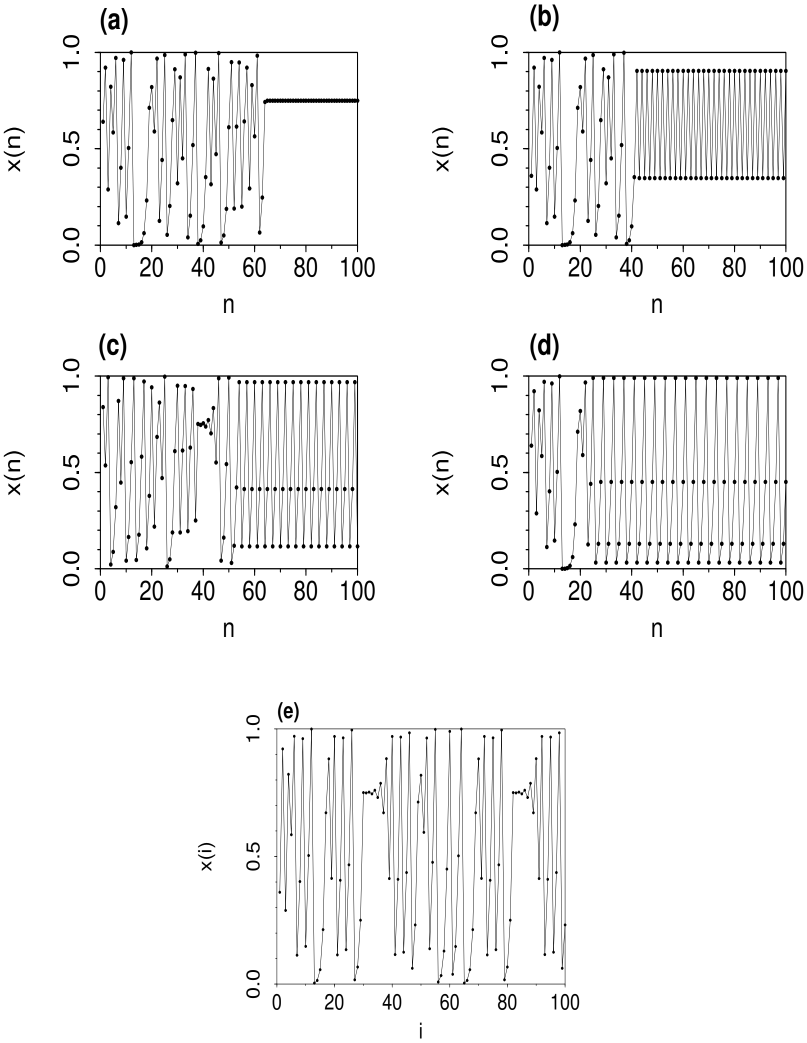

Fig. 2. a-e display examples of orbits for redefined map (1) (see the text) which end up in attractors of periods 1, 2, 3, 4 and 52, respectively.

should be renormalized to $N_0 e^{-\kappa t_s}$, but this does not influence κ.) The definition of escape rate shows that the number of survivors decreases by a factor of $1/e$ after about $1/\kappa$ steps. Therefore, the number $1/\kappa$ is identified with the average lifetime of transients

$$\tau = \frac{1}{\kappa}. \qquad (5)$$

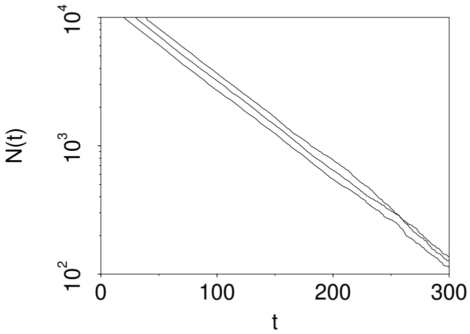

Fig. 3. Distribution of lifetimes for uniformly distributed positions in the interval $[0.07-\varepsilon, 0.07+\varepsilon]$ *for three different values of* $\varepsilon = 10^{-6}$, $\varepsilon = 10^{-9}$ *and* $\varepsilon = 10^{-12}$ *(from left to right) with* $\xi = 0.75$, $d = 0.04$.

On the other hand, the Frobenius-Perron equation leads to

$$\left[1 - \int\limits_{\xi-d/2}^{\xi+d/2} \frac{1}{\pi\sqrt{x\cdot(1-x)}}dx\right]^{\tau} = \frac{1}{e} \tag{6}$$

which gives a semicircle prediction for lifetimes (dashed line in Fig. 4).

The gross behavior of calculated average lifetime is approximated by the Frobenius-Perron equation, but at some positions there appear narrow "bursts" of average lifetime, which sizeably exceed the Frobenius-Perron prediction. To successive groups of peaks (according to the lifetimes) we assign the classification number k so that trajectories starting from initial positions corresponding to a peak of order k escape the unit interval in k iteration steps.

In Table 1 we present the calculated average lifetimes of chaotic transients for peaks corresponding to classification numbers $k = 1$, 2 and 3 for the hole widths $d = 10^{-2}$ and $d = 10^{-4}$. The ratio of average lifetime τ for a peak with classification number k and Frobenius-Perron lifetime (denoted by τ_{FB}) is compared to the estimate

$$R_k = \frac{2^k}{2^k - 1} \tag{7}$$

which was derived in Ref. 28 on the basis of consideration of successive preimages of the hole interval $I^{(0)}$. As seen, the estimate (7) for R_k gives a rather good approximation for the calculated ratio τ/τ_{FB}. We have found that with decrease of the hole with d the average lifetime τ approaches the value $R_k \cdot \tau_{FB}$:

$$\lim_{d\to 0} \tau = R_k \cdot \tau_{FB} \tag{8}$$

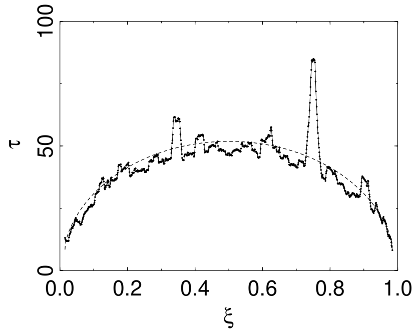

Fig. 4. Average lifetime τ of chaotic transient associated with the map (1) in dependence on the hole position ξ (solid circles). The width of the hole is fixed at $d = 0.03$. Dashed line corresponds to the solutions of Frobenius-Perron equation.

TABLE 1. Average lifetime of chaotic transients for groups of peaks of classification number $k = 1$, 2 and 3 associated with the map (1) for the hole position ξ and the hole width d. τ_{FB} and τ are the average lifetimes determined from the Frobenius-Perron equation and from exponential decay of trajectories starting from 10^6 uniformly distributed initial positions.

k	ξ	τ_{FB}	τ	τ/τ_{FB}		$2^k/(2^k-1)$
				$d = 10^{-2}$	$d = 10^{-4}$	
1	0.750	135.5	264.6	1.95		2.000
2	0.345	148.8	191.4	1.29	1.336	1.333
	0.904	92.0	116.4	1.26	1.335	1.333
3	0.117	100.5	108.8	1.08		1.143
	0.413	154.2	169.2	1.10		1.143
	0.970	52.9	55.6	1.05		1.143
3	0.188	122.2	133.2	1.09		1.143
	0.611	152.7	168.6	1.10		1.143
	0.950	67.9	72.8	1.07		1.143

In order to study the pattern of preimages of $I^{(0)}$ we have calculated the time maps for several peaks of average lifetimes (Fig. 5). If we neglect the preimages

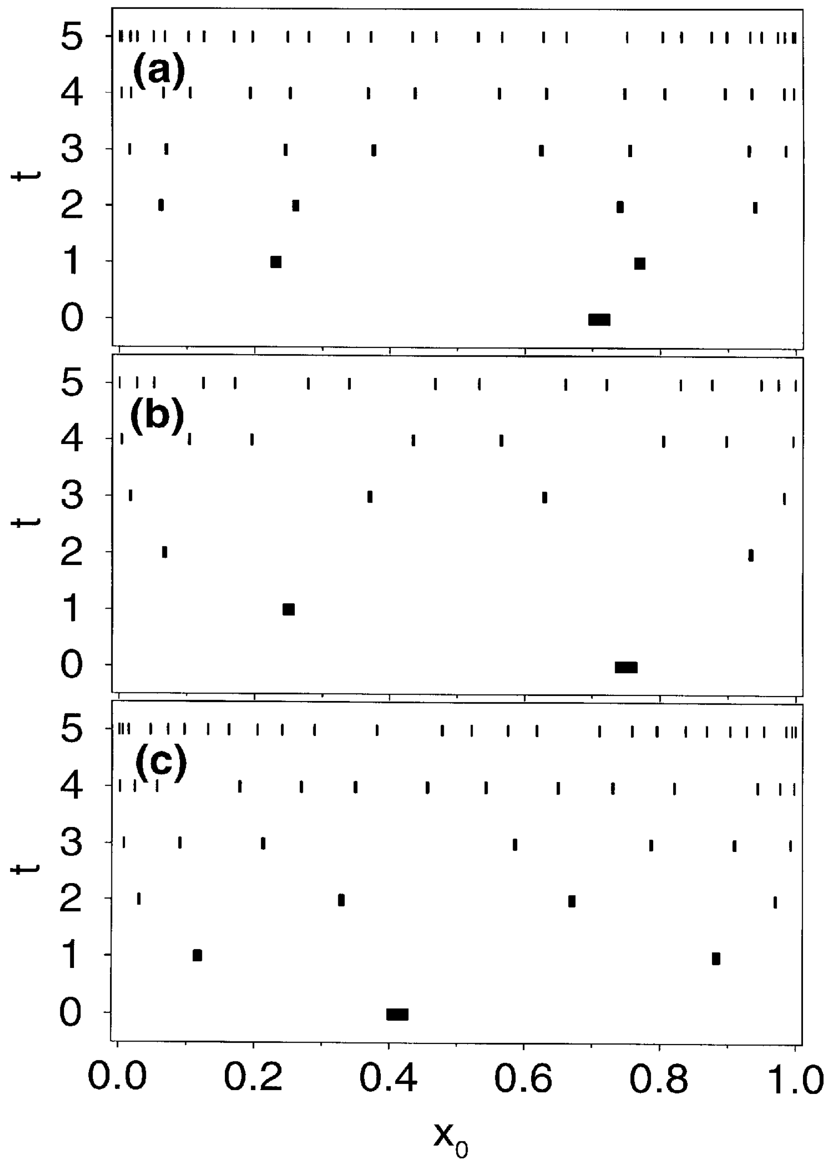

Fig. 5. Time maps for chaotic transients corresponding to the map (1) with the hole position equal to (a) 0.71, (b) 0.75 (c) 0.413. The width of the hole is $d = 0.03$.

which accidentally fall on the interval $I^{(0)}$, i.e. considering the case of the hole width $d \to 0$, we see that for $\xi = 0.5$ (i.e. for a centrally placed hole), the number of preimages in the n-th step is $N_n = 2^n$ (second column in Table 2). Similar result is obtained for $\xi = 0.71$, with the hole lying asymetrically, but outside of positions of pronounced peaks of average lifetime (Fig. 5a).

TABLE 2. Maximum possible number of preimages of the hole interval $I^{(0)}$ in the n-th iteration order determined from the time maps for several low-order peaks of average lifetime.

n	ξ			
	0.5	0.75	0.345	0.414
0	1	1	1	1
1	2	1	2	2
2	4	2	3	4
3	8	4	6	7
4	16	8	12	14
5	32	16	24	28
n	2^n	2^{n-1}	$2^{n-2}(2^2-1)$	$2^{n-3}(2^3-1)$

A different pattern occurs for $\xi = 0.75$, corresponding to the position of largest peak ($k = 0$). In this case $N_0 = 1$, $N_m = 2^{m-1}$ where $m = 1, 2, ...$ (third column in Table 2).

For the $k = 2$ peak at $\xi = 0.345$ we obtain $N_0 = 1$, $N_1 = 2$, $N_m = 2^{m-1}(2^2-1)$ where $m = 2, 3, ...$ (fourth column in Table 2).

For the $k = 3$ peak at $\xi = 0.414$ (Fig. 3c) we obtain $N_0 = 1$, $N_1 = 2$, $N_2 = 4$, $N_m = 2^{m-3}(2^3-1)$ where $m = 3, 4, ...$ (last column in Table 2).

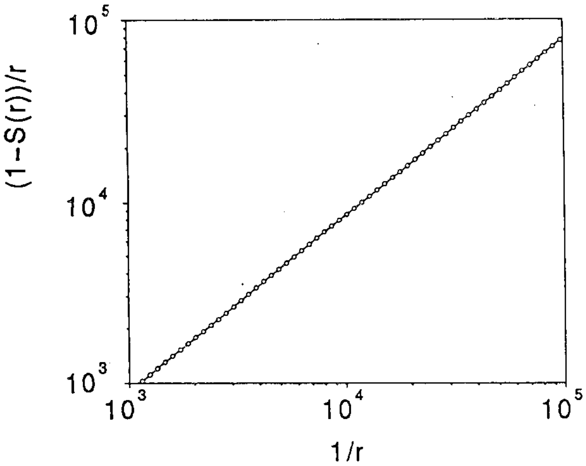

Fig. 6. Determination of fractal dimension of chaotic repeller associated with the map (1) for $\xi = 0.413$, $d = 0.03$ as a slope of the log-log plot of $R(r)$ vs. $1/r$. The slope is $D = 0.9728 \pm 0.0002$.

These results can be combined as

$$N_n = \begin{cases} 2^n, & n < k \\ 2^{n-k}(2^k-1), & n \geq k \end{cases} . \tag{9}$$

Using approximate relations [28]

$$\tau^{-1} \approx \sum_{n=1}^{\infty} \alpha_n N_n \tag{10}$$

$$\tau_{FB}^{-1} \approx \sum_{n=1}^{\infty} \alpha_n 2^n \tag{11}$$

where α_n are certain coefficients, Eq. (9) leads to the approximate relation (8).

3. *Strange repeller associated with the new map*

In a previous study of circle map for the mode-locked intervals the winding number was investigated showing the widths of the steps $\Delta(P/Q)$ versus P/Q [31]. The corresponding graph was a line whose slope gives the fractal dimension. In

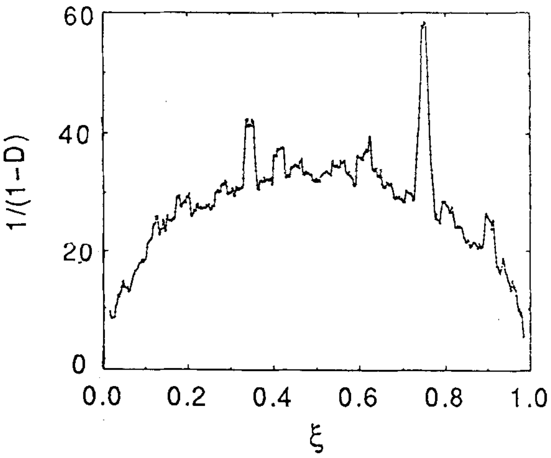

Fig. 7. Fractal dimension D of the repeller in dependence on the hole position ξ *for* $d = 0.03$. *The vertical axis displays* $\frac{1}{1-D}$.

an analog way we have calculated here the total width $S(r)$ of all cylinders which are larger than a given scale r. (According to Ref. 1 the preimage intervals of the unit interval I are called cylinders.) The space between the cylinders, $1 - S(r)$, measured on the scale r gives the number of "holes" $N(r) = [1 - S(r)]/r$. In Fig. 6 the log-log plot of N(r) vs $1/r$ is presented for a set of values of r in the interval $(10^{-3}, 10^{-5})$, and the fractal dimension was determined, as an illustration, for $\xi = 0.413$, $d = 0.03$. The points fall on a straight line indicating the power law

$$N(r) \propto \left(\frac{1}{r}\right)^D . \tag{12}$$

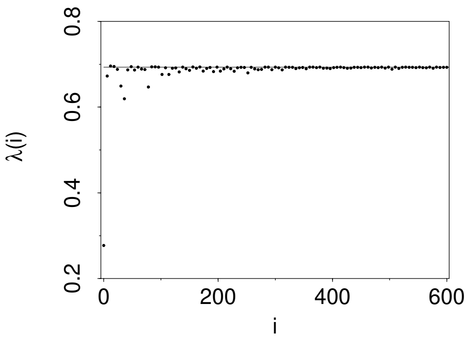

Fig. 8. Lyapunov exponent calculated for the map (1) with $\xi = 0.75$, $d = 0.03$ The result is consistent with $\lambda = \ln 2$ which corresponds to the logistic map.

In this way we obtain $D = 0.9691 \pm 0.0002$ for $\xi = 0.5$ and $D = 0.9829 \pm 0.0002$ for $\xi = 0.75$. Here the uncertainty on D was found from a fit of straight line. In dependence on the hole position ξ the result for fractal dimension of the strange invariant set is displayed in Fig. 7. In accordance with Ref. 28, the graph $\frac{1}{1-D}$ vs ξ coincides with the graph for the average lifetime τ vs ξ. Using the Kantz-Grassberger relation [27]

$$\tau = \frac{1}{\lambda(1-D)} \tag{13}$$

it follows that the Lyapunov exponent λ is a constant, independent on ξ. This prediction is in accordance with directly calculated Lyapunov exponent λ (Fig. 8), which turns out to be practically equal to the value which corresponds to the Lyapunov exponent associated with the $r = 4$ logistic map [27].

4. Discussion and conclusion

Given the cylinders of n-th order, each cylinder expands to the unit interval I in n iterates and so the length Δ of a cylinder is related to the characteristic exponent λ by [32]

$$\Delta = e^{-n\lambda} \tag{14}$$

The entropy $S(\lambda)$ expresses how many cylinders have given λ, i.e. a given length. More precisely, $e^{nS(\lambda)}d\lambda$ is the number of cylinders with characteristic exponent in an interval of size $d\lambda$ around λ. One introduces a partition function $Z_n(\beta)$ as a sum over all cylinders $I_j^{(n)}$ on a given level [32–35].

$$Z_n(\beta) = \sum_j \Delta(I_j^{(n)})^\beta \tag{15}$$

where the parameter β is the analog of inverse temperature in thermodynamics. The growth rate of the partition function defines the pressure $P(\beta)$

$$P(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) \tag{16}$$

which is analog to the free energy density $F(t)$. Here the value $P(1)$ has a simple physical meaning of being equal to $-\kappa$, where κ is the escape rate of points in the unit interval I (Eq. (4)). According to the construction of the cylinders there is [32]

$$\sum_j \Delta(I_j^{(n)}) \sim e^{-n\kappa} \tag{17}$$

from which one obtains the relation to $P(1)$. The sum over cylinders in (15) can be replaced by an integral over characteristic exponents, giving [32]

$$Z_n(\beta) \approx \int e^{nS(\lambda)-n\beta\lambda} d\lambda. \tag{18}$$

Thus, the thermodynamical formalism can be related, in general, to the maps associated with chaotic transients.

Concluding, we have investigated chaotic transients associated with our new map (1) and we deduced an approximate expression for the preimages of the interval defined by the hole. This relation is in accordance with the average lifetimes obtained for the "bursts" with respect to the Frobenius-Perron lifetimes. Finally we note that in the case if the map (1) is modified by defining $f(x) = \frac{1}{2}[f(\xi - \frac{d}{2}) + f(\xi + \frac{d}{2})]$ in the interval $I^{(0)}$, the position of the bursts of average lifetime are associated with stabilization of unstable periodic orbits [29].

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NEDOSTAJUĆE PREDSLIKE ZA KAOTIČNU LOGISTIČKU MAPU S RUPOM

Kaotični tranzijenti i predslike istražuju se za novu mapu koja ima rupu unutar jediničnog intervala logističke mape za $r = 4$. Ovu mapu karakteriziraju odstupanja od Frobenius-Perronove jednadžbe za vrijeme poluraspada u ovisnosti o položaju rupe, u obliku skokova u vremenu poluraspada. Primjenom vremenske mape istražuje se kako ti skokovi nastaju kao posljedica nedostajućih predslika rupnog intervala. Izvodi se približan izraz za omjer vremena poluraspada dobivenog pomoću Frobenius-Perronove i Kantz-Grassbergerove jednadžbe.