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MODIFIED MASSLESS DIRAC EQUATION

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The modified massless Dirac equation introduced in a previous paper is here studied in more detail, especially for transformation properties and invariances. Features which differ from the standard massless case include the absence of a manifest chiral rotation invariance for the full massless spinor (this invariance is concealed by "rescaling" effects), and the existence of a spinor potential.

1. Introduction

This paper is a continuation of a previous one [1], where a new massless Dirac equation was first introduced as a particular case of a modified Dirac formulation. The new equation is here studied in more detail, especially for transformation properties and gauge invariance. The resulting theory is one which is appropriate for massless left-handed neutrinos interacting with (massless) Z^0 gauge particles. The outlined treatment is done before second quantization and before possible symmetry breaking effects [2-4]. In particular, anomalies [3,4] are not a concern (as they only arise after second quantization), nor is a Higgs mechanism [3,4] present in order to give mass to the gauge bosons. As compared to the standard massless Dirac theory, the formulation based on this new equation has several distinctive features. For example, the full neutrino spinor does not display a manifest invariance under chiral rotations (Sect. 5), and its equation is not separately invariant under spatial parity (Sect. 3) and charge conjugation (Sect. 4). Also, the right-handed component of the full spinor needs not be eliminated from the picture, but remains non-interactive and can be interpreted as a spinor potential. The motivation for this study [1] is provided by the interest in distinguishing between strictly massless Dirac fermions and "provisionally" massless ones (the latter being those which are meant to acquire mass through some symmetry breaking effect).

Notation is rather standard throughout the paper, with Greek (Latin) indices running through the values 0,1,2,3 (1,2,3). The summation convention is applied to repeated up and down labels, and units are such that $\hbar = c = 1$.

2. *Modified massless Dirac equation: covariance and gauge invariance*

In a frame of reference $\mathcal{X}$ of real spacetime coordinates $x = \{x^\mu\}$ and pseudoeuclidean metric $g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, the modified massless Dirac equation introduced in a previous paper [1] may be written as follows:

$$\mathrm{i}\gamma^\alpha \partial_\alpha \Psi(x) = \frac{m}{2}(I - \varepsilon\gamma^5)\Psi(x), \qquad m \neq 0. \tag{1}$$

Here, $\Psi(x)$ is a complex four-spinor and the Dirac matrices γ^μ (in a fixed chosen representation) obey the usual rules

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \tag{2}$$

with I being the 4×4 identity matrix and $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Also, $\varepsilon = (-1)^{S+T}$, where S and T indicate the space-index and the time-index [2] of the frame $\mathcal{X}$: namely, $S = 0$ if $s = \{x^k\}$ is a right-handed triplet ($S = 1$ otherwise), and $T = 0$ if $t = x^0$ runs forward ($T = 1$ otherwise). Equation (1) leads to a conserved real current [1]

$$J^\mu(x) = \frac{q}{2}\overline{\Psi}(x)\gamma^\mu(I - \varepsilon\gamma^5)\Psi(x), \qquad \overline{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0, \tag{3}$$

where $q \in \mathbf{R}$ is some appropriate (non-vanishing) charge parameter.
For real linear coordinate transformations of the Poincaré group

$$x'^\mu = \lambda^\mu{}_\alpha x^\alpha + b^\mu, \qquad \lambda^\mu{}_\alpha g^{\alpha\beta}\lambda^\nu{}_\beta = g^{\mu\nu}, \tag{4}$$

and keeping m invariant ($m' = m$), the related spinor transformations can be realized in the same manner of the standard massive Dirac theory [3–11]:

$$\Psi'(x') = \Lambda\Psi(x), \qquad \Lambda^{-1}\gamma^\mu\Lambda = \lambda^\mu{}_\alpha\gamma^\alpha, \qquad \Lambda^\dagger = \pm\gamma^0\Lambda^{-1}\gamma^0, \tag{5}$$

where the positive (negative) sign in Eq. (5) applies to the orthoctronous (antiorthochronous) cases. For each transformation, Λ is only arbitrary by a (global) phase factor and the spinor $\Psi'(x')$ satisfies the equation

$$\mathrm{i}\gamma^{\alpha}\partial'_{\alpha}\Psi'(x') = \frac{m}{2}(I-\varepsilon'\gamma^5)\Psi'(x'). \tag{6}$$

This shows the covariance of the theory. In particular, with convenient choices of the aforementioned phase factors, a spatial inversion ($t' = t, s' = -s$) and a temporal inversion ($t' = -t, s' = s$) will respectively have

$$\Lambda = \mathrm{i}\gamma^0 \quad \text{(for a spatial inversion)}, \qquad \Lambda = \gamma^0\gamma^5 \ \text{(for a temporal inversion)}. \tag{7}$$

If q is taken to be a true scalar, the current $J^{\mu}(x)$ is not a true four-vector under the full Poincaré group of transformations: rather, it exhibits a pseudovectorial behavior under a temporal inversion. Since this is not particularly elegant (although in agreement with similar conventions prevalent in the literature), we here prefer to define [2]:

$$q = (-1)^T q_+, \tag{8}$$

where q_+ is a true scalar (the value of q in a time-forward frame). Thus, with the convention (8) enforced

$$J'^{\mu}(x') \equiv \frac{q'}{2}\overline{\Psi'}(x')\gamma^{\mu}(I-\varepsilon'\gamma^5)\Psi'(x') = \lambda^{\mu}{}_{\alpha}J^{\alpha}(x), \tag{9}$$

for all transformations of type (4).

The phase invariance of Eq. (1) is identical to that of the standard massive Dirac equation [3-11]: each solution $\Psi(x)$ stays a solution (and for the same value or the current) if multiplied by a (global) phase factor $\exp(\mathrm{i}c)$, with $c \in \mathbf{R}$. Thus, Eq. (1) can be made gauge-invariant [3-11] by the minimal replacement (hermitian operator $\to$ hermitian operator):

$$\mathrm{i}\partial_{\mu} \ \to \ \mathrm{i}\mathcal{D}_{\mu} \equiv \ \mathrm{i}\partial_{\mu} - q_+Y_{\mu}(x) = \mathrm{i}\partial_{\mu} - (-1)^T qY_{\mu}(x). \tag{10}$$

This gives

$$\mathrm{i}\gamma^{\alpha}\left[\partial_{\alpha} + \mathrm{i}q_+Y_{\alpha}(x)\right]\Psi(x) = \frac{m}{2}(I-\varepsilon\gamma^5)\Psi(x), \tag{11}$$

which does not change the formal expression (3) of the conserved current. Here, $Y_{\mu}(x)$ is a real vector potential, from which we define the antisymmetric field

$$Z_{\mu\nu}(x) = \partial_{\mu}Y_{\nu}(x) - \partial_{\nu}Y_{\mu}(x). \tag{12}$$

This is taken to satisfy

$$\partial_\alpha Z^{\alpha\mu}(x) = J^\mu(x). \tag{13}$$

The relevant equations (Eqs. (11) and (13), with (12) and (3) as definitions) are easily shown to be covariant under the full Poincaré group (4), with $Y_\mu(x)$ being a true four-vector and with spinorial transformations still given by (5). The current continues to be a true four-vector according to (9), and the field $Z_{\mu\nu}(x)$ behaves like a true four-tensor.

For the sake of clarity, the following point is worth remarking: if the convention had been chosen that q is a true scalar, the transformation properties of $J^\mu(x)$ (as well as those of $Z_{\mu\nu}(x)$ and $Y_\mu(x)$) would be somewhat less elegant. However, the minimal replacement would remain as in the rightmost-hand side of Eq. (10), since $(-1)^T qY_\mu(x)$ would still be a true four-vector. (Although rarely mentioned in the literature, this comment similarly relates to the theory of the standard massive Dirac equation.)

3. Active transformations of the Poincaré group

Unlike the standard massive Dirac theory, the present theory is manifestly non-invariant under spatial parity and charge conjugation. The detailed behavior under active transformations will be examined in this section and the next.

A linear active transformation (of the Poincaré variety) is a change of the field variables in the following manner [3-11]:

$$\widetilde{\Psi}(x) = \Psi'(x), \qquad \widetilde{Y}_\mu(x) = Y'_\mu(x), \tag{14}$$

which keeps m invariant ($\widetilde{m} = m$) and is associated with the definitions

$$\widetilde{Z}_{\mu\nu}(x) = \partial_\mu \widetilde{Y}_\nu(x) - \partial_\nu \widetilde{Y}_\mu(x), \tag{15}$$

$$\widetilde{J}^\mu(x) = \frac{q}{2}\widetilde{\Psi}^\dagger(x)\gamma^0\gamma^\mu(I-\varepsilon\gamma^5)\widetilde{\Psi}(x) \tag{16}$$

$$\widetilde{\mathcal{D}}_\mu = \partial_\mu + \mathrm{i}q_+\widetilde{Y}_\mu(x). \tag{17}$$

For convenience, we also define:

$$\widetilde{K}^\mu(x) = \frac{q}{2}\widetilde{\Psi}^\dagger(x)\gamma^0\gamma^\mu(I+\varepsilon\gamma^5)\widetilde{\Psi}(x). \tag{18}$$

The primed notation in Eq. (14) is taken over from Sect. 2, but no change of coordinates is actually performed. As usual, the invariance of the theory under a transformation of type

(14) is determined by verifying whether the relevant equations for the transformed fields have (or do not have) the same forms as the original ones for the untransformed fields. In our case, invariance is observed for all active transformations corresponding to the proper Poincaré group. The difference with the standard massive Dirac theory becomes evident when we consider a spatial parity transformation

$$\widetilde{\Psi}(x) = \mathrm{i}\gamma^0\Psi(t,-s), \qquad \widetilde{Y}_\mu(x) = Y^\mu(t,-s), \tag{19}$$

for which we obtain

$$\mathrm{i}\gamma^\alpha\widetilde{\mathcal{D}}_\alpha\widetilde{\Psi}(x) = \frac{m}{2}(I+\varepsilon\gamma^5)\widetilde{\Psi}(x), \tag{20}$$

$$\partial_\alpha\widetilde{Z}^{\alpha\mu}(x) = \widetilde{K}^\mu(x). \tag{21}$$

Thus, spatial parity invariance is violated, and this violation is built in the theory even in the absence of interaction ($q_+ \to 0$).

For a temporal parity transformation

$$\widetilde{\Psi}(x) = \gamma^0\gamma^5\Psi(-t,s), \qquad \widetilde{Y}_\mu(x) = -Y^\mu(-t,s), \tag{22}$$

the situation is similar to that described above. Specifically, the transformed field variables (22) satisfy an equation like (20), and the following:

$$\partial_\alpha\widetilde{Z}^{\alpha\mu}(x) = -\widetilde{K}^\mu(x). \tag{23}$$

By comparison, the standard massive Dirac theory is invariant under temporal parity in the limit of a vanishing interaction.

When spatial parity and temporal parity are combined into the spacetime parity transformation

$$\widetilde{\Psi}(x) = -\mathrm{i}\gamma^5\Psi(-x), \qquad \widetilde{Y}_\mu(x) = -Y_\mu(-x), \tag{24}$$

invariance is obtained in Eq. (11), but Eq. (13) is non-invariant:

$$\partial_\alpha\widetilde{Z}^{\alpha\mu}(x) = -\widetilde{J}^\mu(x). \tag{25}$$

As might have been expected, the theory becomes invariant under (24) in the limit of a vanishing interaction.

4. Charge conjugation and time reversal

Charge conjugation can be satisfactorily introduced only after second quantization [10]. However, in the present context, it is still possible to to simulate this transformation. To that end, we first rewrite Eq. (13) as follows:

$$\partial_\alpha Z^{\alpha\mu}(x) = \sigma J^\mu(x), \tag{26}$$

with $\sigma = 1$. Then, we define charge conjugation as the (antilinear) conjugation operation [9]

$$\widetilde{\Psi}(x) = \gamma^5 B \Psi^*(x), \qquad \widetilde{Y}_\mu(x) = -Y_\mu(x), \tag{27}$$

which keeps m invariant ($\widetilde{m} = m$) and is coupled with

$$\widetilde{\sigma} = -\sigma. \tag{28}$$

Here, B is a fixed chosen unitary matrix such that [9]

$$(\gamma^\mu)^* = B^\dagger \gamma^\mu B. \tag{29}$$

Other active transformations related to the conjugation operation may be introduced by composition of Eqs. (14), (27) and (28). For example: time reversal (temporal parity and (27), with no change in σ), PC transformation (charge conjugation and spatial parity), and TPC (PC and time reversal). They are as follows:

$$\widetilde{\Psi}(x) = \gamma^0 B \Psi^*(-t, s), \qquad \widetilde{Y}_\mu(x) = Y^\mu(-t, s), \qquad \widetilde{\sigma} = \sigma \qquad \text{(time reversal)} \tag{30}$$

$$\widetilde{\Psi}(x) = \mathrm{i}\gamma^0\gamma^5 B \Psi^*(t, -s), \qquad \widetilde{Y}_\mu(x) = -Y^\mu(t, -s), \qquad \widetilde{\sigma} = -\sigma \qquad (PC) \tag{31}$$

$$\widetilde{\Psi}(x) = -\mathrm{i}\gamma^5 \Psi(-x), \qquad \widetilde{Y}_\mu(x) = -Y_\mu(-x), \qquad \widetilde{\sigma} = -\sigma \qquad (TPC) \tag{32}$$

With definitions like (15) - (18) extended to the above operations, it is readily seen that the PC transformation leads to:

$$\partial_\alpha \widetilde{Z}^{\alpha\mu}(x) = \widetilde{\sigma}\widetilde{J}^\mu(x) \tag{33}$$

$$i\gamma^{\alpha}\widetilde{\mathcal{D}}_{\alpha}\widetilde{\Psi}(x) = \frac{m^*}{2}(I - \varepsilon\gamma^5)\widetilde{\Psi}(x). \tag{34}$$

The same happens for time reversal. Hence, invariance is guaranteed for both PC and time reversal if m is real (see also Sect. 5). For TPC, invariance is obtained even for complex m. On the other hand, charge conjugation leads to the following:

$$\partial_{\alpha}\widetilde{Z}^{\alpha\mu}(x) = \widetilde{\sigma}\widetilde{K}^{\mu}(x) \tag{35}$$

$$i\gamma^{\alpha}\widetilde{\mathcal{D}}_{\alpha}\widetilde{\Psi}(x) = \frac{m^*}{2}(I + \varepsilon\gamma^5)\widetilde{\Psi}(x). \tag{36}$$

These equations display a lack of invariance which persists in the limit of a vanishing interaction, irrespective of the real or complex character of m.

In concluding this section, we remark that the adopted phase convention for the Λ's of Sect. 2 conforms to the equation

$$\Lambda^* = \pm B^{\dagger}\Lambda B, \tag{37}$$

where the positive (negative) sign is valid for the Poincaré transformations with positive (negative) determinant. This provides that $\Psi(x)$ and its charge conjugate behave in identical manners under Poincaré transformations [3-11]; it also reduces the arbitrariness of the Λ's to the sign.

5. Other invariances?

According to [1], the value of the non-vanishing parameter m is physically irrelevant. Hence, one should consider the possibility of rescaling: that is, allowing $m' = r'm \neq 0$ for coordinate transformations, and $\widetilde{m} = \widetilde{r}m \neq 0$ for active transformations. This would generalize the invariances and transformations already discussed in the previous sections. For instance, a generalized global "phase" invariance for Eq. (1) would involve an operation of the type

$$\widetilde{\Psi}(x) = U\Psi(x), \tag{38}$$

leaving Eq. (1) invariant in form (but with possible rescaling)

$$i\gamma^{\alpha}\partial_{\alpha}\widetilde{\Psi}(x) = \frac{\widetilde{m}}{2}(I - \varepsilon\gamma^5)\widetilde{\Psi}(x), \tag{39}$$

and leaving the current invariant in value

$$\widetilde{J}^{\mu}(x) = J^{\mu}(x). \tag{40}$$

Equations (39) and (40) are obtainable (with Eqs. (1) and (39) being equivalent) if the matrix U satisfies

$$U^{\dagger}\gamma^{0}\gamma^{\mu}(I-\varepsilon\gamma^{5})U = \gamma^{0}\gamma^{\mu}(I-\varepsilon\gamma^{5}), \qquad \det(U) \neq 0 \tag{41}$$

$$U^{\dagger}\gamma^{\mu}U = \widetilde{r}\gamma^{\mu} + W\gamma^{\mu}(I-\varepsilon\gamma^{5}), \qquad \widetilde{r} = \frac{\widetilde{m}}{m} \neq 0, \tag{42}$$

where W is an arbitrary 4×4 matrix.

No-rescaling solutions of the type $U = \exp(\mathrm{i}c)I$ (with $c \in \mathbf{R}$) were introduced in Sect. 2. These are the only solutions of Eqs. (41) and (42), corresponding to $\widetilde{r} = 1$ and were gauged (i.e. made local) by means of the minimal replacement (10). For values $\widetilde{r} \neq 1$, other solutions exist which are not multiples of the identity matrix and are not necessarily unitary: all are linear combinations of I and γ^{5}. For instance, it is worth mentioning the chiral rotations $U = \exp(\mathrm{i}d\varepsilon\gamma^{5})$ (with $d \in \mathbf{R}$), leading to $\widetilde{r} = \exp(2\mathrm{i}d)$. However, unless the rescaling solutions are gauged, they may as well be ”disabled” in the present context, by choosing a value of m (real, due to the existence of antilinear operations[1]) and keeping it unchanged under all transformations. For comparison, we remind that the standard massive Dirac equation also possesses a rescaling invariance, but much more limited, related to the sign ambiguity of its (real) mass parameter: it is usually eliminated by selecting the positive value for the mass parameter [6].

Here, we do not wish to explore the possibility of gauging the rescaling invariance. Thus, in the light of the previous discussion, rescaling is excluded altogether: $m' = \widetilde{m} = m \in \mathbf{R} - \{0\}$. Furthermore, we notice that it is convenient to identify m with the (positive) mass parameter of the massive counterpart to Eq. (1). In other words, if Eq. (1) applies to electron neutrinos, m is taken to be the electron mass, etc. [1].

6. *Spinor potential*

The introduction of the right-handed component $\Psi_R(x)$ and left-handed component $\Psi_L(x)$ of $\Psi(x)$ [1] shows that Eq. (11) can be equivalently rewritten as the two equations

$$\mathrm{i}\gamma^{\alpha}\mathcal{D}_{\alpha}L(x) = 0 \tag{43}$$

$$L(x) = \mathrm{i}\gamma^{\alpha}\mathcal{D}_{\alpha}R(x), \tag{44}$$

[1] If m is not chosen to be real, Eq. (1) is not invariant under PC or time reversal as defined in Sect. 4, but could be made invariant by means of rescaling (making use of an appropriate chiral rotation).

where

$$L(x) \equiv \Psi_L(x), \qquad R(x) \equiv \Psi_R(x)/m, \tag{45}$$

while the current (3) is given by

$$J^\mu(x) = q\overline{L}(x)\gamma^\mu L(x). \tag{46}$$

As suggested in [1], $R(x)$ can be viewed as a spinor potential for the the physically relevant quantity $L(x)$. The theory is then reduced to two definitions of physically relevant fields from potentials (Eqs. (12) and (44)), and two equations for the physically relevant fields (Eqs. (13) with the current (46), and Eq. (43)). It is easily seen that $L(x)$ remains invariant in value if $R(x)$ undergoes the change:

$$R(x) \to R(x) + \overset{\circ}{R}(x), \tag{47}$$

where $\overset{\circ}{R}(x)$ satisfies the equation

$$\mathrm{i}\gamma^\alpha \mathcal{D}_\alpha \overset{\circ}{R}(x) = 0. \tag{48}$$

7. *Conclusions*

This paper, together with [1], proposes the use of a modified massless Dirac equation for strictly massless left-handed Dirac fermions. Properties of this new equation provide a consistent framework for its use in the electroweak theory. The four-component standard massless Dirac equation would still be useful for Dirac fermions (e.g., electrons) which are meant to acquire a mass after the introduction of some symmetry breaking effect [1].

The new massless Dirac equation has some very interesting features, including the possibility of rescaling and the existence of a spinor potential. Besides, it solves the standing problem [10] of the "superfluous" right-handed current.

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MODIFICIRANA DIRACOVA JEDNADŽBA ZA ČESTICE BEZ MASE

U ranijem je radu izvedena modificirana Diracova jednadžba za čestice bez mase, a ovdje se podrobnije proučavaju posebice njena transformacijska svojstva i invarijantnost. Njene posebitosti, koje je razlikuju od standardne bezmasene jednadžbe, su odsustvo manifestne kiralne rotacijske invarijantnosti potpunog bezmasenog spinora (ta je invarijantnost sakrivena učincima *"rescalinga"* i pojava spinornog potencijala.