ISSN 1330–0008
CODEN FIZAE4

EXACT SOLUTIONS FOR THE GENERALIZED VARIABLE-COEFFICIENT KdV EQUATION[1]

YAN-ZE PENG

Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China
e-mail address: yanzepeng@163.com

Received 16 March 2003; Accepted 21 July 2002
Online 20 September 2003

The generalized variable-coefficient Korteveg de Vries equation is investigated by means of the polynomial expansion method and many travelling-wave solutions are constructed, including the solitary wave, the triangular periodic wave and the rational solutions. The technique is straightforward to use and only minimal algebra is needed to find these solutions. A simple review of the method is finally given.

PACS numbers: 02.30.Jr, 05.45.Yv UDC 531.435, 531.327.13

Keywords: generalized variable-coefficient KdV equation, travelling-wave solution, polynomial expansion method

1. *Introduction*

There are many methods for obtaining special solutions of a nonlinear partial differential equation (PDE). Some of the most important methods are the inverse scattering transformation (IST) [1], the bilinear method [2], symmetry reductions [3] and Bäcklund and Darboux transformations [4]. Recently, directly searching for the exact solutions of the nonlinear PDEs has become more and more attractive for their important role in understanding the nonlinear phenomena. One of the most effectively straightforward methods to construct the exact solution of PDEs is tanh-function method [5–9]. However, this method can only be used to deal with the nonlinear PDEs with constant coefficients while the assumption of constant coefficients is highly idealized. So far, direct methods for obtaining exact solutions of nonlinear PDEs with variable coefficients are rare, although Chan and Zheng [10] and Chan and Li [11] studied the non-isospectral and variable-coefficient Korteveg

[1]This work is supported by the Postdoctoral Science Foundation of China

de Vries (KdV) equation by the methods of Bäcklund transformations and inverse scattering. Therefore, it is important to develop a simple and direct method for getting exact solutions of nonlinear variable-coefficient PDEs, which is one of the purpose of our paper.

2. The method

In this article, a polynomial expansion method is proposed to obtain the exact solutions of the generalized variable-coefficient KdV equation. The basic idea of the method is as follows. For a given PDE with variable coefficients, say in two variables,

$$P(t, x, u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1}$$

we seek for the solutions of Eq. (1) of the form

$$u(x, t) = \sum_{n=0}^{m} A_n(t) w^n, \tag{2}$$

where $w \equiv w(\xi)$ satisfies

$$w' = w^2 + pw + q\,. \tag{3}$$

Here the prime denotes the derivation with respect to ξ (throughout the paper), p, q are parameters, $\xi = f(t)x + g(t)$, $f(t)$ and $g(t)$ are functions to be determined. m in Eq. (2) is a positive integer that can be determined by balancing the linear term of the highest order with the nonlinear term in Eq. (1). Substituting Eq. (2) into Eq. (1) and using Eq. (3) will yield a set of differential equations and algebraic equations for $A_i(t)$, $f(t)$, $g(t)$, p and q, because all coefficients of w^i must vanish. From these relations, $A_i(t)$, $f(t)$, $g(t)$, p and q may be determined. In general, if any of the parameters and/or functions is left unspecified, then it is to be regarded as arbitrary for the solution of Eq. (1). If we assume $w = \tanh\xi$ instead of Eq. (3), our method is a trivial generalization of Malfliet's [5]. Very recently, Fan and Zhang [12] obtained the doubly-periodic wave solutions for some special-type nonlinear PDEs by means of the Jacobi elliptic function method with symbolic computation. Under the limiting conditions, these doubly-periodic wave solutions degenerate to the solitary wave or the shock wave or the triangular periodic-wave solutions. However, they did not deal with the nonlinear PDEs with variable coefficients. We choose Eq. (3) because the shock wave $f = \tanh\xi$, the periodic wave $f = \tan\xi$, etc., are all solutions of it for appropriate values of the parameters p and q. Moreover, the formal solution (2) is a polynomial in w and the derivatives of u can also be expressed as the polynomials in w through Eq. (3). It is due to the entrance of two parameters, p and q in Eq. (3), that we may obtain multiple travelling-wave solutions of the equation under consideration. This technique is straightforward to use and only minimal algebra is needed to find these solutions.

3. Application

In this section, we study the generalized variable-coefficient KdV equation

$$u_t + 2\beta(t)u + [\alpha(t) + \beta(t)x]u_x - 3c\gamma(t)uu_x + \gamma(t)u_{xxx} = 0. \tag{4}$$

The non-isospectral and variable-coefficient KdV equation [10-11, 13-14], generalized KdV equation and the cylindrical KdV equation [13], etc., are all special forms of Eq. (4). Substituting Eq. (2) with $m = 2$ into Eq. (4) and using Eq. (3) yields a set of equations (after equating the coefficients of the like powers of w to zero)

$$w^5: \quad -6c\gamma A_2^2 f + 24\gamma A_2 f^3 = 0, \tag{5}$$

$$w^4: \quad -9c\gamma A_1 A_2 f + 6\gamma A_1 f^3 - 6c\gamma p A_2^2 f + 54\gamma p A_2 f^3 = 0, \tag{6}$$

$$\begin{aligned} w^3: \quad & 2A_2(f_t x + g_t) + 2(\alpha + \beta x)A_2 f - 6c\gamma A_0 A_2 f \\ & -3c\gamma A_1^2 f - 6c\gamma A_2^2 q f + 40\gamma A_2 q f^3 \\ & -9c\gamma p A_1 A_2 f + 12\gamma p A_1 f^3 + 38\gamma p^2 A_2 f^3 = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} w^2: \quad & A_{2t} + A_1(f_t x + g_t) + 2\beta A_2 + (\alpha + \beta x)A_1 f \\ & -3c\gamma A_0 A_1 f - 9c\gamma A_1 A_2 q f + 8\gamma A_1 q f^3 \\ & +2pA_2(f_t x + g_t) + 2p(\alpha + \beta x)A_2 f - 6c\gamma p A_0 A_2 f - 3c\gamma p A_1^2 f \\ & +7\gamma p^2 A_1 f^3 + 52\gamma p q A_2 f^3 + 8\gamma p^3 A_2 f^3 = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} w^1: \quad & A_{1t} + 2A_2 q(f_t x + g_t) + 2\beta A_1 + 2(\alpha + \beta x)A_2 q f \\ & -6c\gamma A_0 A_2 q f - 3c\gamma A_1^2 q f + 16\gamma A_2 q^2 f^3 \\ & +p(f_t x + g_t)A_1 + p(\alpha + \beta x)A_1 f - 3c\gamma p A_0 A_1 f + \gamma p^3 A_1 f^3 \\ & +14\gamma p^2 q A_2 f^3 + 8\gamma p q A_1 f^3 = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} w^0: \quad & A_{0t} + A_1 q(f_t x + g_t) + 2\beta A_0 + (\alpha + \beta x)A_1 q f \\ & -3c\gamma A_0 A_1 q f + 2\gamma A_1 q^2 f^3 + \gamma p^2 q A_1 f^3 + 6\gamma p q^2 A_2 f^3 = 0. \end{aligned} \tag{10}$$

From Eqs. (5) – (10) follows that

$$A_0(t) = c_0 \exp\Big[-\int 2\beta(t)\mathrm{d}t\Big], \tag{11}$$

$$A_1(t) = \frac{4}{c} p f^2(t), \quad A_2(t) = \frac{4}{c} f^2(t), \tag{12}$$

$$f(t) = c_1 \exp\Big[-\int \beta(t)\mathrm{d}t\Big], \tag{13}$$

$$\begin{aligned} g(t) = \int \big[& -8q\gamma(t)f^3(t) + 3c\gamma(t)A_0(t)f(t) \\ & -\alpha(t)f(t) - \gamma(t)p^2 f^3(t)\big]\mathrm{d}t + c_2, \end{aligned} \tag{14}$$

where c_0, c_1 and c_2 are arbitrary constants. So we obtain the exact solutions of Eq. (4) as follows

$$u(x,t) = A_0(t) + \frac{4}{c} p f^2(t) w + \frac{4}{c} f^2(t) w^2, \tag{15}$$

where w satisfies Eq. (3), with $A_0(t)$, $f(t)$ and $g(t)$ given by Eqs. (11), (13) and (14), respectively, and c_i $(i = 0, 1, 2)$, p, q are arbitrary constants. To be specific, we have several types of exact solutions of Eq. (4) given in what follows.

Case 1. $q = 0, \quad p \neq 0$. Eq. (3) has the solution

$$w = \frac{p}{C_0 \exp(-p\xi) - 1}, \tag{16}$$

where C_0 is an integrating constant. Thus the corresponding solution for Eq. (4) reads

$$u(x,t) = \begin{cases} A_0(t) + \frac{1}{c} f^2(t) p^2 \operatorname{csch}^2 \left[\frac{1}{2}(-p\xi + \ln C_0)\right], & \text{for} \quad C_0 > 0, \\ A_0(t) - \frac{1}{c} f^2(t) p^2 \operatorname{sech}^2 \left[\frac{1}{2}(-p\xi + \ln |C_0|)\right], & \text{for} \quad C_0 < 0. \end{cases} \tag{17}$$

Case 2. $q \neq 0, \quad p = 0$. According to the sign of the constant q, we have subcases to discuss.
As $q < 0$, we obtain soliton-like solutions to Eq. (4)

$$u_1(x,t) = A_0(t) - \frac{4q}{c} f^2(t) \tanh^2 \sqrt{-q}\ [f(t)x + g(t)], \tag{18}$$

$$u_2(x,t) = A_0(t) - \frac{4q}{c} f^2(t) \coth^2 \sqrt{-q}\ [f(t)x + g(t)]. \tag{19}$$

As $q > 0$, we have travelling-wave solutions

$$u_3(x,t) = A_0(t) + \frac{4q}{c} f^2(t) \tan^2 \sqrt{q}\ [f(t)x + g(t)], \tag{20}$$

$$u_4(x,t) = A_0(t) + \frac{4q}{c} f^2(t) \cot^2 \sqrt{q}\ [f(t)x + g(t)]. \tag{21}$$

Case 3. $q = 0, \quad p = 0$. We have rational-type travelling-wave solution

$$u_5(x,t) = A_0(t) + \frac{4 f^2(t)}{c[f(t)x + g(t) + c_3]^2}, \tag{22}$$

where c_3 is an arbitrary constant.

Case 4. $q \neq 0, \quad p \neq 0$. Eq. (3) has the solution

$$w = \frac{w_1 - w_2 c_4 \exp[(w_1 - w_2)\xi]}{1 - c_4 \exp[(w_1 - w_2)\xi]}, \tag{23}$$

where c_4 is an arbitrary constant, and w_1, w_2 are two roots of the equation $w^2 + pw + q = 0$. From Eq. (15), we obtain soliton-like solutions of Eq. (4) as follows

(1) When $c_4 > 0$,

$$u(x,t) = A_0(t) - \frac{4}{c} q f^2(t) + \frac{1}{c} f^2(t) \mathrm{csch}^2 \frac{1}{2}[(w_1 - w_2)\xi + \ln c_4]. \tag{24}$$

(2) When $c_4 < 0$,

$$u(x,t) = A_0(t) - \frac{4}{c} q f^2(t) - \frac{1}{c} f^2(t) \mathrm{sech}^2 \frac{1}{2}[(w_1 - w_2)\xi + \ln |c_4|]. \tag{25}$$

4. *Conclusions and discussion*

We have proposed a polynomial expansion method and used it to solve the generalized variable-coefficient KdV equation. The exact travelling-wave solutions obtained in this paper include soliton solutions, periodic solutions, rational solutions and singular solutions. This method is readily applicable to a large variety of nonlinear PDEs with variable coefficients. Above all, it can be used to construct multiple travelling-wave solutions for nonlinear variable-coefficient PDEs in a unified way. If the balancing number m is not a positive integer, we can introduce a transformation $u = v^{1/m}$ and turn Eq. (1) into another equation for v, whose balancing number will be a positive integer. Then it can be dealt with by the method above. If the equation under consideration is not a differential polynomial type equation, we may introduce an appropriate transformation to turn the equation in question into the differential polynomial type and then deal with it using the described method.

References

[1] C. S. Gardner, J. M. Green, M. D. Kruskal and R. M. Miura, Phys. Rev. Lett. **19** (1967) 1095.

[2] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.

[3] P. J. Olver, *Application of Lie Group to Differential Equations*, Springer, New York (1986).

[4] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).

[5] W. Malfliet, Am. J. Phys. **60** (1992) 650.
[6] E. J. Parkes, J. Phys. A: Math. Gen. **27** (1994) L497.
[7] E. J. Parkes and B. R. Duffy, Comp. Phys. Commun. **98** (1996) 288.
[8] B. R. Duffy and E. J. Parkes, Phys. Lett. A **214** (1996) 271.
[9] E. J. Parkes and B. R. Duffy, Phys. Lett. A **229** (1997) 217.
[10] W. L. Chan and Y. K. Zheng, Lett. Math. Phys. **14** (1987) 293.
[11] W. L. Chan and K. S. Li, J. Math. Phys. **30** (1989) 2521.
[12] E. G. Fan and J. Zhang, Phys. Lett. A **305** (2002) 383.
[13] C. Tian, J. Phys. A: Math. Gen. **20** (1987) 359.
[14] W. L. Chan and X. Zhang, J. Phys. A: Math. Gen. **28** (1995) 407.

EGZAKTNA RJEŠENJA POOPĆENE KdV JEDNADŽBE S PROMJENLJIVIM KOEFICIJENTIMA

Istražili smo poopćenu KdV jednadžbu s promjenljivim koeficijentima metodom razvoja po polinomima. Izvode se mnoga rješenja za putujuće valove, uključivši solitone, trokutne periodičke valove i racionalna rješenja. Postupak je izravan i rješenja se dobivanju vrlo jednostavnim računom. Izlaže se jednostavan pregled postupka.