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# IMPROVED TRUNCATED EXPANSION METHOD AND NEW EXACT SOLUTIONS OF THE GENERAL VARIABLE-COEFFICENT KdV EQUATION

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In this paper, by using a new special function transform and truncated expansion method, three kinds of exact solutions of the general variable-coefficient KdV equation have been obtained. The solutions are general and they contain some exact analytical solutions, which have been given in other papers.

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## 1. *Introduction*

In recent decades, the study of nonlinear problems has been greatly intensified in many areas of science and technology. Most nonlinear problems are characterized by nonlinear equations. The methods of their solution play an important role in the understanding nonlinear problems. Many methods have been proposed, such as the inverse scattering method [1], Backlund transformations method [2], Hirota transformations method [3], Darboux transformations method [4], the homogeneous balance method [5], the hyperbolic expansion method [6], sine-cosine method [7], direct reductions method [8] and Jacobian elliptic function method [9]. However, these methods can only solve the problems of constant-coefficient nonlinear equations. But constant-coefficient nonlinear equations can only characterize approximately the reality of physical phenomena, and studying of the corresponding variable-coefficient nonlinear equations is very important.

In Ref. [10], the general variable-coefficient KdV equation has been studied by the function transform in truncated expansion method

$$u_t + 2\beta(t)u + \big[\alpha(t) + \beta(t)x\big]u_x - 3c\gamma(t)uu_x + \gamma(t)u_{xxx} = 0\,. \tag{1}$$

New kinds of solitary wave solutions were obtained. This method is very efficient.

In this paper, the truncated expansion method is improved and the special function transform is applied. Many kinds of new exact solutions of the general variable-coefficient KdV equation are obtained, which include the new kinds of solitary wave solutions of Ref. [10].

## 2. The improved method

Our improved method can be summed up as follows. For a given general variable-coefficient nonlinear equation

$$U(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots) = 0\,, \tag{2}$$

we seek for its solutions in the form

$$u(x,t) = \sum_{n=0}^{N} A_n(t) F^n, \quad F = F(\xi)\,, \tag{3}$$

where

$$\xi \;=\; f(t)x + g(t) \tag{4}$$

and $f(t)$ and $g(t)$ are some functions which should be found. In solving the nonlinear equation, the process will be very brief if we can transfer from the form of derivatives to the form of a single derivative. So, we propose that $F = F(\xi)$ satisfy

$$F_\xi \;=\; p + qF + rF^2, \tag{5}$$

where $p$, $q$ and $r$ are constants.

When $q^2 < 4pr$, by integrating Eq. (5) with respect to $\xi$ one obtains

$$F = \frac{\sqrt{4pr - q^2}}{2r}\ \mathrm{tg}\,\frac{\sqrt{4pr - q^2}}{2}(\xi + C_1) - \frac{q}{2r}\,. \tag{6}$$

When $q^2 = 4pr$, by integrating Eq. (5) with respect to $\xi$ we obtain

$$F = -\frac{q}{2r} - \frac{1}{r\xi + C_2}\,. \tag{7}$$

When $q^2 > 4pr$, by integrating Eq. (5) with respect to $\xi$ we obtain

$$F = \frac{\sqrt{q^2 - 4pr}}{r} \cdot \frac{C_3\,\mathrm{e}^{\sqrt{q^2 - 4pr}\xi}}{1 - C_3\,\mathrm{e}^{\sqrt{p^2 - 4qr}\xi}} + \frac{\sqrt{q^2 - 4pr} - q}{2r}\,, \tag{8}$$

where $C_1$, $C_2$ and $C_3$ are integration constants. Substitution of Eq. (6), (7) or (8) into Eq. (1) yields the three kinds of exact solutions of the general variable-coefficient KdV equation.

## 3. *The general variable-coefficient KdV equation*

Equating the highest-order derivative term and the nonlinear term in Eq. (1), we obtain $N = 2$, so

$$u = A_0(t) + A_1(t)F + A_2(t)F^2 \tag{9}$$

and we get

$$\begin{aligned} u_t &= A_{0t} + pA_1\xi_t + (A_{1t} + qA_1\xi_t + 2pA_2\xi_t)F \\ &\quad +(rA_1\xi_t + A_{2t} + 2qA_2\xi)F^2 + 2rA_2\xi_tF^3, \end{aligned} \tag{10}$$

$$u_x = \xi_x\big[pA_1 + (qA_1 + 2pA_2)F + (rA_1 + 2qA_2)F^2 + 2rA_2F^3\big], \tag{11}$$

$$\begin{aligned} u_{xxx} &= \xi_x^3\big[pq^2A_1 + 2p^2rA_1 + 6p^2qA_2 \\ &\quad +(q^3A_1 + 8pqrA_1 + 14pq^2A_2 + 16p^2rA_2)F \\ &\quad +(7q^2rA_1 + 8pr^2A_1 + 52pqrA_2 + 8q^3A_2)F^2 \\ &\quad +(12qr^2A_1 + 38q^2rA_2 + 40pr^2A_2)F^3 \\ &\quad +(6r^3A_1 + 54qr^2A_2)F^4 + 24r^3A_2F^5\big]. \end{aligned} \tag{12}$$

Substituting (9) – (12) into Eq. (1) and making that the coefficients of all powers of $F$ are equal to zero, we get

$$F^5: \quad -6c\gamma(t)\xi_x rA_2^2 + 24\gamma(t)\xi_x^3 r^3A_2 = 0, \tag{13}$$

$$F^4: \quad -3c\gamma(t)\xi_x(3rA_1A_2 + 2qA_2^2) + \gamma(t)\xi_x^3(6r^3A_1 + 54qr^2A_2) = 0, \tag{14}$$

$$\begin{aligned} F^3: \quad & 2rA_2\xi_t + 2r[\alpha(t) + \beta(t)x]\xi_xA_2 - 3c\xi_x(2rA_0A_2 + rA_1^2 + 3qA_1A_2 + 2pA_2^2) \\ & \quad +\gamma(t)\xi_x^3(12qr^2A_1 + 38q^2rA_2 + 40pr^2A_2) = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} F^2: \quad & rA_1\xi_t + A_{2t} + 2qA_2\xi_t + 2\beta(t)A_2 + [\alpha(t) + \beta(t)x]\xi_x(rA_1 + 2qA_2) \\ & \quad -3c\gamma(t)\xi_x(rA_0A_1 + 2qA_0A_2 + qA_1^2 + 3pA_1A_2) \\ & \quad +\gamma(t)\xi_x^3(7q^2cA_1 + 8pr^2 + 8q^2A_2 + 52pqrA_2) = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} F^1: \quad & A_{1t} + qA_1\xi_t + 2pA_2\xi_t + 2\beta(t)A_1 + [\alpha(t) + \beta(t)x]\xi_x(qA_1 + 2pA_2) \\ & \quad -3c\gamma(t)\xi_x(qA_0A_1 + 2pA_0A_2 + pA_1^2) \\ & \quad +\gamma(t)\xi_x^3(q^3A_1 + 8pqrA_1 + 14pq^2 + 16p^2rA_2) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} F^0: \quad & A_{0t} + pA_1\xi_t + 2\beta(t)A_0 + [\alpha(t) + \beta(t)x]\xi_xpA_1 - 3c\gamma(t)\xi_xpA_0A_1 \\ & \quad +\gamma(t)\xi_x^3(pq^2A_1 + 2p^2rA_1 + 6p^2qA_2) = 0. \end{aligned} \tag{18}$$

From Eqs. (13) and (14) it follows that

$$A_2 = \frac{4r^2}{c}\xi_x^2 = \frac{4r^2}{c}\ f^2(t), \qquad A_1 = \frac{4qr}{c}\xi_x^2 = \frac{4qr}{c}\ f^2(t)\,. \tag{19}$$

From Eqs. (15)–(17) and (19) it follows that

$$\xi_t + [\alpha(t) + \beta(t)x]\, f(t) - r\gamma(t)A_0\, f(t) + (q^2 + 8pr)\gamma\, f^3(t) = 0\,. \tag{20}$$

This proves that the former hypothesis is self-consistent. Comparing (20) to (5), we obtain

$$\begin{aligned} f_t &= -\beta(t)\, f(t), \\ g_t &= -\alpha(t)\, f(t) + 3c\gamma(t)A_0\, f(t) - (q^2 + 4pr)\gamma(t)\, f^3(t)\,. \end{aligned} \tag{21}$$

Substituting (20) into (18), we obtain

$$A_{0t} + 2\beta(t)A_0 = 0\,. \tag{22}$$

From Eqs. (21) and (22), we get

$$\begin{aligned} f(t) &= C_f\, \mathrm{e}^{-\int \beta(t)\mathrm{d}t}, \\ g(t) &= \int \big[ -\alpha(t)\, f(t) + 3c\gamma(t)A_0\, f(t) - (q^2 + 8pr)\gamma(t)\, f^3(t)\big]\, \mathrm{d}t + C_g, \\ A_0 &= C_0\, \mathrm{e}^{-\int 2\beta(t)\mathrm{d}t} \end{aligned} \tag{23}$$

where $C_f$, $C_g$ and $C_0$ are integration constants.

So we can get the exact analytic solution of Eq. (1),

$$\begin{aligned} u &= A_0 + \frac{4qr}{c} f^2(t)F(\xi) + \frac{4r^2}{c} f^2(t)F^2(\xi) \\ &= A_0 + \frac{4r}{c} f^2(t)\, [qF(\xi) + rF^2(\xi)]\,. \end{aligned} \tag{24}$$

When constants $p$, $q$ and $r$ have different values, we can also obtain new kinds of exact solutions of variable-coefficient KdV equation. When $q^2 < 4pr$,

$$u = \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{C_f^2 q^2}{c} + \frac{(4pr - q^2)C_f^2}{c}\ \mathrm{tg}^2\, \frac{\sqrt{4pr - q^2}}{2}(\xi + C_1)\right]. \tag{25}$$

If $\sqrt{4pr - q^2}/2 = 1$, Eq. (25) takes the form

$$\begin{aligned} u &= \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{C_f^2 q^2}{c} + \frac{4C_f^2}{c}\ \mathrm{tg}^2(\xi + C_1)\right] \\ &= \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{C_f^2 q^2}{c}\right] + \mathrm{e}^{-2\int \beta \mathrm{d}t}\frac{4C_f^2}{c}\ \mathrm{tg}^2(\xi + C_1). \end{aligned} \tag{26}$$

This is in consonance with the second exact analytic solution of Ref. [11].

When $q^2 = 4pr$, we obtain

$$u = \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{4prC_f^2}{c} + \frac{4r^2C_f^2}{c}\frac{1}{(r\xi + C_2)^2}\right]. \tag{27}$$

When $q^2 > 4pr$,

$$u = \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{4prC_f^2}{c} + \frac{4(q^2-4pr)C_f^2}{c}\cdot\frac{C_3\mathrm{e}^{\sqrt{q^2-4pr}\xi}}{\left(1 - C_3\mathrm{e}^{\sqrt{q^2-4pr}\xi}\right)^2}\right]. \tag{28}$$

From Eq. (28), if $\sqrt{q^2-4pr}/2 = 1$, $C_3 = -1$, we obtain

$$\begin{aligned} u &= \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{4prC_f^2}{c} - \frac{16C_f^2}{c}\cdot\frac{\mathrm{e}^{2\xi}}{(1+\mathrm{e}^{2\xi})^2}\right] \\ &= \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{4prC_f^2}{c} + \frac{4C_f^2}{c}\right] + e^{-2\int \beta \mathrm{d}t}\frac{4C_f^2}{c}\ \tanh^2\xi. \end{aligned} \tag{29}$$

This is in consonance with the first exact analytical solution of Ref. [11].

From Eq. (28), if $p = 0$, $q = -1$, $r = 1$, $C_3 = -1$, we get

$$\begin{aligned} u &= \mathrm{e}^{-2\int \beta \mathrm{d}t}\left[C_0 - \frac{4C_f^2}{c}\cdot\frac{\mathrm{e}^{\xi}}{(1+\mathrm{e}^{\xi})^2}\right] \\ &= C_0\,\mathrm{e}^{-2\int \beta \mathrm{d}t} - \mathrm{e}^{-2\int \beta \mathrm{d}t}\frac{C_f^2}{c}\ \mathrm{sech}^2\frac{1}{2}\xi. \end{aligned} \tag{30}$$

This is in consonance with the soliton solution (34) of Ref. [10].

## 4. *Conclusions*

In this paper, we have presented a new function transform form and obtained new kinds of exact solutions of variable-coefficient KdV equation with improved truncated expansion method, which include the exact solutions of Ref. [10]. In addition, when the constants $p$, $q$ and $r$ take up different values, we can obtain many new exact solutions. This method of solving equations is adapted to solving other variable-coefficient nonlinear equations.

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## POBOLJŠANA METODA PREKIDA RAZVOJA I NOVA RJEŠENJA OPĆE KdV JEDNADŽBE S PROMJENLJIVIM KOEFICIJENTIMA

U ovom radu, primjenom posebnih pretvorbi funkcija i prekidom razvoja, postigli smo tri egzaktna rješenja opće KdV jednadžbe s varijabilnim koeficijentima. Postignuta rješenja su općenita i sadrže neka poznata analitička rješenja u drugim radovima.