Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

COUPLED OSCILLATORS: AN INFORMATIVE PROBLEM SOLVING APPROACH

CARLOS HUNTE

University of The West Indies, Cave Hill Campus Department of Computer Science, Mathematics and Physics, P.O. Box 64, Bridgetown, Barbados, West Indies

Received 14 March 2007; Accepted 7 November 2007
Online 8 February 2008

A simple and informative method of solving for the normal modes and the normal mode frequencies of coupled oscillating systems is presented. A dimensionless parameter uniquely determines the normal modes and normal mode frequencies of the oscillating system.

PACS numbers: 01.40.-d, 05.45.Xt, 46.40.-f UDC 534.014.1
Keywords: coupled oscillators, normal modes, normal mode frequencies

1. Introduction

All mechanical and electrical vibration problems reduce, in the case of small oscillations, to problems involving one or several coupled oscillators. Problems involving vibrations of strings, membranes, elastic solids, electrical or acoustical can be reduced to problems of coupled oscillators. The effect of coupling in a simple system with two degrees of freedom (DOF) produces two characteristic frequencies and two normal modes of oscillation. The general motion of the system is a superposition of the normal modes of oscillation, but initial conditions can always be found so that any one of the normal modes can be independently excited. Identifying each of a systems normal modes allows a construction of a complete picture of the motion, even though the general motion of the system is a complicated combination (superposition) of all normal modes.

Various methods are used to solve for these normal modes: The Lagrangian method, the Hamiltonian method and the Newton's force method, to name a few. Of these, only the Newton's force method is suitable for use in an introductory or intermediate course on classical mechanics, where the concept of force is of fundamental importance. In the Newton's force method, the standard way of solving [1 - 5] is to:

(i) assume oscillatory solutions of the displacements;

(ii) substitute the solutions in *(i)* into the equations of motion;

(iii) solve a system of equations (usually in the form of a secular determinant) to determine the normal mode frequencies;

(iv) determine the normal modes of oscillation.

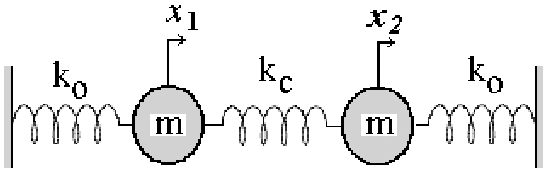

Fig. 1. Coupled oscillators, the problem regularly treated in textbooks on general physics.

For comparison of the proposed method to the standard method, we present the latter as can be found in many textbooks. The equations of motion are obtained by considering the forces on each mass separately. Let the masses be displaced by amounts x_1 and x_2 from their equilibrium position as shown in Fig. 1. The equations of motion for the two masses are

$$m\ddot{x}_1 = -k_0 x_1 - k_c(x_1 - x_2)\,, \tag{1}$$

$$m\ddot{x}_2 = -k_0 x_2 - k_c(x_2 - x_1)\,, \tag{2}$$

where $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$.

Adding and subtracting Eqs. (1) and (2) gives

$$(\ddot{x}_1 + \ddot{x}_2) = -\frac{k_0}{m}(x_1 + x_2)\,, \tag{3}$$

$$(\ddot{x}_1 - \ddot{x}_2) = -\frac{k_0 + 2k_c}{m}(x_1 - x_2)\,, \tag{4}$$

Eqs. (3) and (4) represent in-phase SHM of the two masses with angular frequency $\sqrt{k_0/m}$ and anti-phase SHM of the two masses with angular frequency $\sqrt{(k_0 + 2k_c)/m}$, respectively.

Here, $(x_1 + x_2)$ and $(x_1 - x_2)$ are the normal mode coordinates of the oscillating system.

In this article, an alternative method of solving, based on Newtons force method, is presented. This algebraic method reduces the amount of calculation needed to determine the normal modes of vibration and normal mode frequencies. The normal modes of vibration are uniquely determined by the parameter λ. By forcing the resulting normal mode equations to resemble those of simple harmonic motion (SHM), the underlying physics is borne out clearly in this method of solving. Since the method is not widely known, this article could serve to introduce the method to instructors of introductory classical mechanics and, thereby, encourage them to cover the topic of coupled oscillators in their courses.

We begin by examining a system exhibiting 2 DOF. The method is then extended to two systems having 3 DOF. This method is general and can be applied to any system having n degrees of freedom. For a system having n degrees of freedom, introduction of $n-1$ dimensionless parameters is needed and one must form a suitable linear combination of the relative displacements. The coupled systems shown in Figs. 1−4 are used to illustrate the method. By forming a suitable combination of the equations of motion, we determine that the resulting motion is oscillatory, and then easily determine the normal modes and normal mode frequencies.

2. *General method*

The above method of solving takes advantage of the symmetry of the mass-spring system. However, when there is no symmetry, a general method must be used to solve for the normal modes. This method consists of introducing a dimensionless parameter λ, and forming the combination $(\ddot{x}_1 + \lambda\ddot{x}_2)$ of the equations of motion of the masses. The parameter λ describes the motion of the mass 2 relative to the mass 1.

If we apply this technique to the problem above, we expect to get $\lambda = \pm 1$ and hence arrive at the same solutions for the normal mode frequencies. When we form the combination $(\ddot{x}_1 + \lambda\ddot{x}_2)$, we get

$$\begin{aligned}(\ddot{x}_1 + \lambda\ddot{x}_2) &= x_1(\lambda k_C - k_C - k_0) + x_2(k_C - \lambda k_C - \lambda k_0)/m \\ &= -\left(\frac{k_0 + k_C - \lambda k_C}{m}\right)\left[x_1 + \left(\frac{k_C - \lambda k_C - \lambda k_0}{\lambda k_C - k_C - k_0}\right)x_2\right].\end{aligned} \tag{5}$$

Let X represent the quantity $x_1 + \lambda x_2$. Then Eq. (5) can be written as $\ddot{X} = -\omega^2 X$. Here X is the normal mode coordinate of the system.

This represents SHM with angular frequencies given by

$$\omega^2 = \left(\frac{k_0 + k_C - \lambda k_C}{m}\right) \tag{6}$$

and

$$\lambda = \left(\frac{k_C - \lambda k_C - \lambda k_0}{\lambda k_C - k_C - k_0}\right). \tag{7}$$

One solution of this equation is $\lambda = +1$, and the masses move in phase with angular frequency given by $\omega_+^2 = k_0/m$. The other solution is $\lambda = -1$, the masses move in anti-phase with angular frequency $\omega_+^2 = (k_0 + 2k_C)/m$, as expected.

Thus we recover the two normal mode frequencies with this method. The parameter λ determines the normal modes of the system.

We next apply this method to a mass-spring system that lacks the symmetry of the above problem.

Problem 2

Consider the mass-spring system shown in Fig. 2a.

The equations of motion for the two masses are given by:

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2), \tag{8}$$

$$m\ddot{x}_2 = -k(x_2 - x_1)\,. \tag{9}$$

Eqs. (8) and (9) are to be solved to determine the normal mode frequencies of the oscillating system. Note that the gravitational forces acting on the masses need not be considered here since they are independent of the displacements and hence do not contribute to the restoring forces that cause the oscillations. The gravitational forces merely cause a shift in the equilibrium positions of the masses [1].

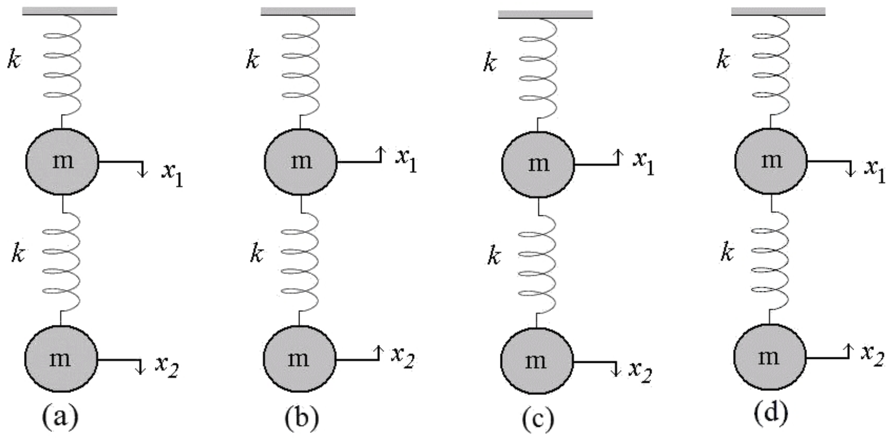

Fig. 2. System of two masses hung on springs. Equations of motion, (Eqs. (8) and (9) in the text) are formulated assuming that x_1 and x_2 are displacements from the equlibrium positions as shown in figure a). Figures a) and b) illustrate the mode when the two masses oscillate in phase with the parameter $\lambda_+ = (1+\sqrt{5})/2$. Figures c) and d) illustrate the mode when the two masses oscillate with opposite phase with the parameter $\lambda_- = (1-\sqrt{5})/2$.

To solve, we again introduce the parameter λ into the coupled motion to form a linear combination of the displacements x_1 and x_2. Thus, we form the combination and simplify to give

$$(\ddot{x}_1 + \lambda\ddot{x}_2) = \omega_0^2[(\lambda - 2)x_1 + (1 - \lambda)x_2] \tag{10}$$

i.e.,

$$(\ddot{x}_1 + \lambda\ddot{x}_2) = -\omega_0^2(2-\lambda)\left[x_1 + \left(\frac{1-\lambda}{\lambda-2}\right)x_2\right], \tag{11}$$

where $\omega_0^2 = k/m$. Hence, the combined motion is clearly simple harmonic with angular frequency ω, given by

$$\omega^2 = \omega_0^2(2-\lambda) \tag{12}$$

and

$$\lambda = \frac{1-\lambda}{\lambda-2}. \tag{13}$$

It is now justified to assume oscillatory solutions of x_1 and x_2. Equation (12) determines the normal mode frequencies of oscillation of the coupled system. Solving Eq. (13) gives

$$\lambda^2 - \lambda - 1 = 0, \tag{14}$$

with the solutions $\lambda_+ = (1+\sqrt{5})/2$ and $\lambda_- = (1-\sqrt{5})/2$.

There are thus two normal modes of the system described by λ_+ and λ_-, corresponding to the in-phase and anti-phase oscillations of the two masses, respectively.

Substituting λ_+ and λ_- into Eq. (12) gives the normal mode frequencies (eigenfrequencies) of the oscillations

$$\omega_\pm^2 = \omega_0(2-\lambda_\pm) = \omega_0^2\left(\frac{3\mp\sqrt{5}}{2}\right). \tag{15}$$

Figures 2a and 2b show the first normal mode of the oscillating system. Here the masses oscillate in phase, i.e., both masses move in the same direction. In this mode $\lambda = \lambda_+ = \frac{1}{2}(1+\sqrt{5})$ and $x_2 = \frac{1}{2}(1+\sqrt{5})x_1$. The steady-state amplitude of x_2 is always greater that that of x_1, and the system oscillates with lower frequency.

In the second mode shown in Figs. 2c and 2d, the phase difference of oscillations of the two masses is equal to π, i.e., the masses move in opposite directions. In this mode $\lambda = \lambda_- = \frac{1}{2}(1-\sqrt{5})$ and $x_2 = \frac{1}{2}(1-\sqrt{5})x_1$. The amplitude of x_2 is always less than that of x_1, and the system vibrates with the higher frequency.

Problem 3

We now extend the method to a system having 3 DOF. Systems with a larger number of degrees of freedom are treated similarly.

Consider the system shown in Fig. 3. We need two dimensionless parameters, say λ_2 and λ_3, representing the ratios of the amplitudes of x_2 to x_1 and of x_3 to x_1, respectively.

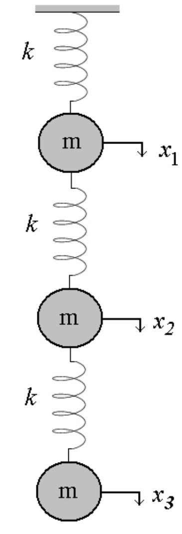

Fig. 3. System of three masses hung on springs. Equations of motion (Eqs. (16), (17) and (18) in the text) are formulated assuming that x_1, x_2 and x_3 are displacements from the equlibrium positions. The figure also shows the first mode when all three masses oscillate in phase.

The equations of motion for the three masses are

$$m\ddot{x}_1 = -2kx_1 + kx_2 \,, \tag{16}$$

$$m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 \,, \tag{17}$$

$$m\ddot{x}_3 = kx_2 - kx_3 \,. \tag{18}$$

We now form the combination $(\ddot{x}_1 + \lambda_2\ddot{x}_2 + \lambda_3\ddot{x}_3)$. Forming this combination and simplifying gives

$$(\ddot{x}_1+\lambda_2\ddot{x}_2+\lambda_3\ddot{x}_3) = -\omega_0^2(2-\lambda_2)\left[x_1+\left(\frac{1-2\lambda_2+\lambda_3}{\lambda_2-2}\right)x_2+\left(\frac{\lambda_2-\lambda_3}{\lambda_2-2}\right)x_3\right]. \tag{19}$$

Now, we introduce replacements

$$\omega^2 = \omega_0^2(2-\lambda_2), \tag{20}$$

where

$$\lambda_2 = (1-2\lambda_2+\lambda_3)/(\lambda_2-2)\,, \tag{21}$$

and

$$\lambda_3 = (\lambda_2-\lambda_3)/(\lambda_2-2)\,. \tag{22}$$

Combining Eqs. (21) and (22) gives

$$\lambda_2^3 - \lambda_2^2 - 2\lambda_2 + 1 = 0\,. \tag{23}$$

Using MATLAB to solve this cubic equation gives the following solutions: $\lambda_2 =$ 1.8019, -1.2470 and 0.4450. Hence we can calculate the corresponding values of λ_3 and of ω. These values are tabulated in Table 1.

TABLE 1. Values of λ_2 with corresponding values of λ_3 and angular frequency ω.

λ_2	1.8019	-1.247	0.4450
λ_3	2.2470	0.5550	-0.8018
ω/ω_0	0.445	1.8019	1.247

In one mode (first column) the masses move with the same phase (all three masses move in the same direction). It is the mode with lowest frequency ω and both λ_2 and λ_3 are positive. In the other modes two of the three masses move in the same direction and one in the opposite direction as either λ_2 or λ_3 are negative.

Problem 4

Consider the ring structure shown in Fig. 4a which consists of three equal masses m which slide without friction on a circular ring of radius R. The masses are connected by identical springs of spring constant k. The angular positions of the three masses are measured from a rest position.

For small displacements from equilibrium, (assuming equal masses) the equations of motion can be written as

$$\ddot{\theta}_1 = -\omega_0^2(2\theta_1 - \theta_2 - \theta_3)\,, \tag{24}$$

$$\ddot{\theta}_2 = -\omega_0^2(2\theta_2 - \theta_3 - \theta_1)\,, \tag{25}$$

$$\ddot{\theta}_3 = -\omega_0^2(2\theta_3 - \theta_1 - \theta_2)\,, \tag{26}$$

where $\omega_0^2 = k/m$.

Thus

$$(\ddot{\theta}_1+\lambda_2\ddot{\theta}_3+\lambda_3\ddot{\theta}_3)=-\omega_0^2(2-\lambda_2-\lambda_3)\left[\theta_1+\left(\frac{2\lambda_2-\lambda_3-1}{2-\lambda_2-\lambda_3}\right)\theta_2+\left(\frac{2\lambda_3-\lambda_2-1}{2-\lambda_2-\lambda_3}\right)\theta_3\right]. \tag{27}$$

This equation represents SHM with angular frequency given by

$$\omega^2 = \omega_0^2(2 - \lambda_2 - \lambda_3) \tag{28}$$

if

$$\lambda_2 = \frac{2\lambda_2 - \lambda_3 - 1}{2 - \lambda_2 - \lambda_3} \tag{29}$$

and

$$\lambda_3 = \frac{2\lambda_3 - \lambda_2 - 1}{2 - \lambda_2 - \lambda_3} \tag{30}$$

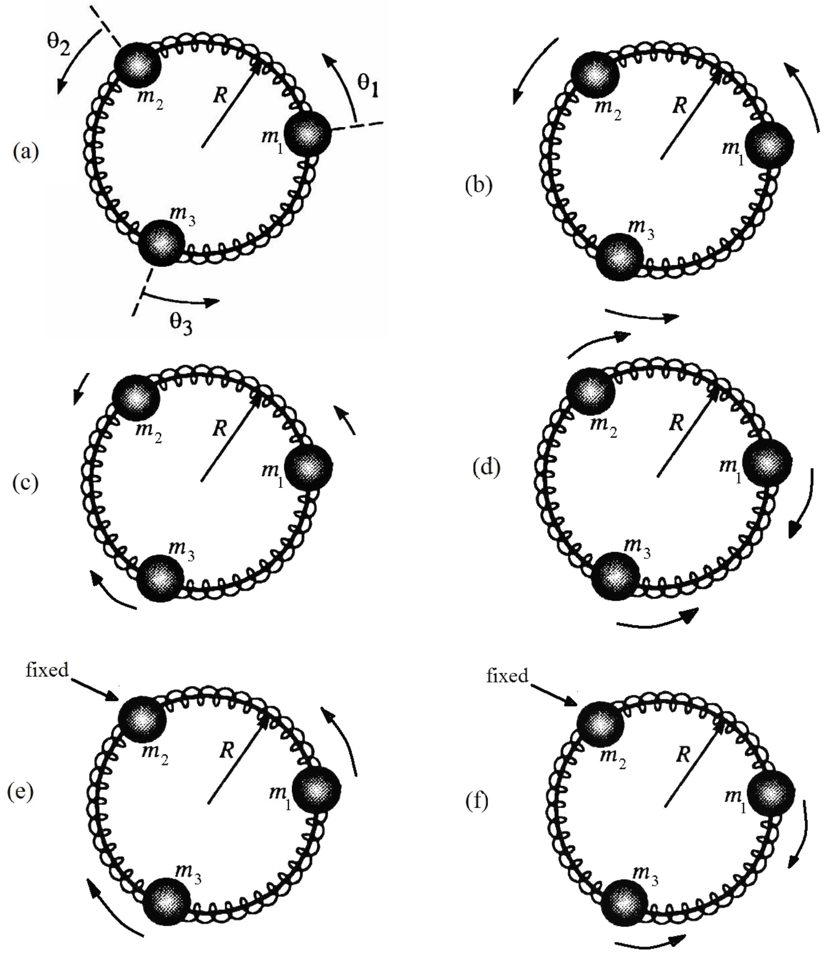

Fig. 4. System of three masses on a circular ring with elastic springs between them. a) The positions are given by angles θ_1, θ_2 and θ_3, measured from a fixed direction and motion is given by Eqs. (24), (25) and (26) in the text. b) The first solution is a simple rotation, i.e., no oscillations are present. c) and d) In the second solution, masses m_1 and m_2 oscillate in phase and mass m_3 with opposite phase. e) and f) In the third solution, one mass is stationary (m_2), while the other two masses oscillate with opposite phase.

It is easy to find solutions of Eq. (29): $\lambda_2 = 1$ and $\lambda_2 = -(1 + \lambda_3)$ and of Eq. (30): $\lambda_3 = 1$ and $\lambda_3 = -(1 + \lambda_2)$. Hence, three modes are possible:

(i) $\lambda_2 = \lambda_3 = 1$ when $\omega^2 = \omega_0^2(2 - \lambda_2 - \lambda_3) = 0$;

(ii) $\lambda_2 = 1$ and $\lambda_3 = -(1-\lambda_2)$ when $\omega^2 = \omega_0^2(2-\lambda_2-\lambda_3) = 3\omega_0^2$; and

(iii) $\lambda_2 = -(1-\lambda_3)$ and $\lambda_3 = -(1-\lambda_2)$ when $\omega^2 = \omega_0^2(2-\lambda_2-\lambda_3) = 3\omega_0^2$.

Hence the normal mode frequencies are given by $\omega = 0$, $\omega = \sqrt{3}\omega_0$ and $\omega = \sqrt{3}\omega_0$. Notice that there is a degeneracy between two of the normal modes with frequency $\sqrt{3}\omega_0$.

Consider the mode *(i)*, $\lambda_2 = \lambda_3 = 1$ and $\omega^2 = \omega_0^2(2-\lambda_2-\lambda_3) = 0$. For this zero-frequency mode, all three masses move at constant and equal velocity around the ring as shown in Fig. 4b, so there are no oscillations at all and all springs remain at their equilibrium lengths.

In the mode *(ii)*, the masses m_1 and m_2 oscillate in phase with the same amplitude as shown in Figs. 4c and 4d. The spring between the two masses remains at its equilibrium length. However, the mass m_3 oscillates in anti-phase with the amplitude twice as large as the amplitude of either m_1 or m_2.

Consider the mode *(iii)*, $\lambda_2 = -(1-\lambda_3)$, $\lambda_3 = -(1-\lambda_2)$ and $\omega^2 = \omega_0^2(2-\lambda_2-\lambda_3) = 3\omega_0^2$. Since $\lambda_2 = -(1-\lambda_3)$, then $\theta_1+\theta_2+\theta_3 = 0$. If θ_2 is arbitrarily set to zero, (m_2 held stationary), then $\theta_1 = -\theta_3$ and m_1 and m_3 oscillate in anti-phase with the same amplitude. If θ_3 is set to zero (m_3 held stationary), then and m_1 and m_2 oscillate in anti-phase with the same amplitude. In these cases one of the coordinates is constant while the other two coordinates oscillate with opposite phase as shown in Figs. 4e and 4f.

3. Conclusions

The coupled harmonic oscillator, traditionally reserved for upper-level physics courses, is studied at a level appropriate for introductory college physics students. This suggests that other important systems, usually considered too complicated for the introductory physics class, are in fact very suitable when studied with modern techniques.

An alternative approach of solving for the normal modes and normal mode frequencies of coupled oscillators with two or more degrees of freedom is presented. The introduction of the symmetry parameter reduces the amount of calculation needed to obtain the normal mode frequencies of vibration of the coupled system. The method described can be extended to any number of coupled mechanical or electrical harmonic oscillators. The algebraic details become tedious for systems with large numbers of degrees of freedom and we often have to resort to numerical methods in order to solve such systems. However, this method of solving has the advantage over other general methods of solving in that requires much less algebraic manipulations. There are no assumptions about the motion of the masses other than that they are displaced by small amounts from equilibrium. The method of solving requires no calculus and is hence suitable for both majors and non majors in physics.

It is hoped that with this method of solving, introductory physics students can study more interesting and realistic problems than they could previously, and this can only increase their understanding of and hopefully their enthusiasm for physics.

References

[1] A. P. French, *Vibrations and Waves*, Nelson, Cheltenham (1971) p. 156.

[2] T. W. B. Kibble, F. H. Berkshire, *Classical Mechanics*, Longman, Essex (1996) p. 207.

[3] K. R. Symon, *Mechanics*, 3e, Addison Wesley, Reading, Massachusetts (1971) p. 193.

[4] H. J. Pain, *The Physics of Vibrations and Waves*, 2e, Wiley, Chichester (1976), p. 73.

[5] J. B. Marion, S. T. Thornton, *Classical Dynamics of Particles and Systems*, 4e, Saunders College Publishing, Philadelphia (1995) p. 461.

VEZANI OSCILATORI: POUČAN NAČIN RJEŠAVANJA

Izlaže se jednostavan i poučan način rješavanja problema vezanih oscilatora radi dobivanja svojstvenih modova i frekvencija. Uvodi se bezdimenzijski parametar koji jednoznačno određuje svojstvene načine i frekvencije oscilatornog sustava.