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JACKSON'S PARADOX AND ITS RESOLUTION BY THE FOUR-DIMENSIONAL GEOMETRIC QUANTITIES

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In this paper it is shown that the real cause of Jackson's paradox is the use of three-dimensional (3D) quantities, e.g., $\mathbf{E}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{L}$, $\mathbf{T}$, their transformations and relations. The principle of relativity is naturally satisfied and there is no paradox when the physical reality is attributed to the 4D geometric quantities, e.g., to the 4D torque N (bivector) or, equivalently, to the 4D torques N_s and N_t (1-vectors), which together contain the same physical information as the bivector N.

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1. *Introduction*

In a recent paper [1], Jackson discovered an apparent paradox: there is a three-dimensional (3D) torque and so a time rate of change of 3D angular momentum in one inertial frame, but no 3D angular momentum and no 3D torque in another relatively moving inertial frame. (In this paper the paradox examined in Ref. [1] will be called the Jackson's paradox.) Two inertial frames S (the laboratory frame) and S' (the moving frame) are considered in Ref. [1] (they are K and K', respectively, in Jackson's notation). In S' a particle of charge q and mass m experiences only the radially directed electric force, $\mathbf{F}' = q\,\mathbf{E}'$, caused by a point charge Q fixed permanently at the origin. Hence, in S', the subsequent motion of the particle is outward along the original line making an angle θ' with the x' axis. This means that the 3D vectors $\mathbf{r}'$, $\mathbf{p}'$ and $\mathbf{F}'$ point in the same direction. Consequently both $\mathbf{L}'$, $\mathbf{L}' = \mathbf{r}'\times\mathbf{p}'$, and the torque $\mathbf{T}'$, $\mathbf{T}' = \mathbf{r}'\times\mathbf{F}'$, (Jackson's $\mathbf{N}'$ is denoted as $\mathbf{T}'$) are zero in S', see Fig. 1(a) in Ref. [1]. In that figure the charges are considered at time $t' = 0$ when the particle of charge q is released at rest with $r'(0) = r'_0$. (The vectors in the 3D space will be designated in bold-face.)

Then, in Ref. [1], the same system is considered from the frame S at $t' = t = 0^+$. The S' frame moves uniformly in the x direction with $v = c\beta$ in S. (Jackson's v_0, β_0 are denoted as v, β.) Hence, relative to S the charge Q is in uniform motion and it produces *both* an electric field $\mathbf{E}$ and a *magnetic field* $\mathbf{B}$. This situation, at the initial time, is shown in Fig. 1(b) in Ref. [1]. The existence of $\mathbf{B}$ in S is responsible for the existence of the 3D magnetic force $\mathbf{F}_{\mathrm{M}}(\mathbf{0}) = q\,\beta \times \mathbf{B}$, which points in the negative y direction and this force provides a 3D torque $\mathbf{T}$ on the charged particle. Consequently a nonvanishing angular momentum of the charged particle changes in time in S, $\mathbf{T} = \mathrm{d}\mathbf{L}/\mathrm{d}\mathbf{t}$.

Here we repeat Jackson's words [1] about such result: "How can there be a torque and so a time rate of change of angular momentum in one inertial frame, but no angular momentum and no torque in another? Is there a paradox? Some experienced readers will see that there is no paradox - *that is just the way things are*, ..." (my emphasis)

In order to "prove" that there is no paradox, Jackson [1] derived in two ways that T_z in S is different from zero. First, in Sec. II, the torque T_z is directly obtained from the force equation in the laboratory, see Eq. (7) in Ref. [1]. Then, in Sec. III, the common transformations, which are usually considered to be the Lorentz transformations (LT) (boosts), of the components of the 3D angular momentum $\mathbf{L}$ are written, Eq. (11) in Ref. [1], here (2), and it is shown that L_z in S is different from zero, Eq. (13) in Ref. [1], i.e., that, [1]: "the angular momentum does not vanish in the laboratory frame K, even if it does in K'." (As already mentioned, in our notation K and K' are denoted as S and S', respectively.) Furthermore, using so obtained L_z, it is shown that $\mathrm{d}L_z/\mathrm{d}t$, Sec. III in Ref. [1], is equal to the torque T_z that is found in Sec. II in Ref. [1], i.e., that [1] "The time rate of change of the particle's angular momentum obtained via a Lorentz transformation is equal to the torque directly obtained from the force equation in the laboratory, as it must." Because of that Jackson [1] considers that there is no paradox and that such result is relativistically correct result, i.e., that the principle of relativity is not violated. In our opinion the paradox remained completely untouched in the approach from Ref. [1]. What is found in Ref. [1] is that when the 3D vectors and their transformations (like Eq. (11) in Ref. [1], here (2)) are used then the paradox is always obtained and, actually, *the principle of relativity is violated*.

In this paper, as in Ref. [2], it will be first shown that the usual transformations of the components of the 3D vector $\mathbf{L}$, which are given by Eq. (11) in Ref. [1], here (2), are not the LT. Furthermore, it will be shown, as in Ref. [2], that the real cause of the paradox is - the use of 3D quantities, e.g., $\mathbf{E}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{L}$, $\mathbf{T}$, their transformations and relations. In Ref. [2], instead of using 3D quantities, it is dealt from the outset with 4D geometric quantities. In that treatment, the paradox does not appear and the principle of relativity is naturally satisfied. In this paper we shall briefly repeat some parts of the consideration from Ref. [2], but now in the form which is better suited for students and teachers. It is worth noting that exactly the same paradox appears in the Trouton-Noble experiment, see, e.g., Ref. [3] and references therein.

2. A brief summary of geometric algebra

The calculations in Ref. [2] (also in Ref. [3]) and in this paper are performed in the geometric algebra formalism. Physical quantities will be represented by 4D geometric quantities, multivectors, that are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, they are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis. For simplicity and for easier understanding only the standard basis, described below, will be used.

Here, for readers' convenience, we provide a brief summary of geometric algebra. Usually Clifford vectors are written in lower case (a) and general multivectors (Clifford aggregate) in upper case (A). The space of multivectors is graded and multivectors containing elements of a single grade, r, are termed homogeneous and usually written A_r. The geometric (Clifford) product is written by simply juxtaposing multivectors AB. A basic operation on multivectors is the degree projection $\langle A \rangle_r$ which selects from the multivector A its $r-$ vector part (0 = scalar, 1 = vector, 2 = bivector ....). The geometric product of a grade-r multivector A_r with a grade-s multivector B_s decomposes into $A_r B_s = \langle AB \rangle_{\,r+s} + \langle AB \rangle_{\,r+s-2} ... + \langle AB \rangle_{\,|r-s|}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series; $A_r \cdot B_s \equiv \langle AB \rangle_{\,|r-s|}$ and $A_r \wedge B_s \equiv \langle AB \rangle_{\,r+s}$. For vectors a and b we have: $ab = a \cdot b + a \wedge b$, where $a \cdot b \equiv (1/2)(ab + ba)$, $a \wedge b \equiv (1/2)(ab - ba)$.

In this paper, the notation will not be the same as in the above mathematical presentation. Namely some 1-vectors will be denoted in lower case, like u, v (the velocities), x (the position 1-vector), p (the proper momentum), while some others in upper case, like the 1-vectors of the electric and magnetic fields E and B respectively, the torques N_s and N_t, the angular momenta M_s and M_t, the Lorentz force K_L. Bivectors will be denoted in upper case but without subscript that denotes the grade. Thus the electromagnetic field F, the torque N and the angular momentum M are all the bivectors.

In, e.g., Ref. [4], one usualy introduces the standard basis. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors $\{\gamma_\mu\}, \mu = 0, \ldots, 3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -)$. This basis, the standard basis, is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone. The γ_k ($k = 1, 2, 3$) are spacelike vectors. The γ_μ generate by multiplication a complete basis for the spacetime algebra: 1, γ_μ, $\gamma_\mu \wedge \gamma_\nu$, $\gamma_\mu \gamma_5$, γ_5 ($2^4 = 16$ independent elements). γ_5 is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. For all mathematical details regarding the spacetime algebra reader can consult Ref. [4]. It is worth noting that the standard basis $\{\gamma_\mu\}$ corresponds, in fact, to the specific system of coordinates, i.e., to Einstein's system of coordinates. In Einstein's system of coordinates, the standard, i.e., Einstein's synchronization [5] of distant clocks and Cartesian space coordinates x^i, are used in the chosen inertial frame. However, different systems of coordinates are allowed in an inertial frame and they are all equivalent in the description of physical phenomena. For example, in Ref. [6] two very different, but physically completely equivalent, systems of coordinates, Einstein's system of coordinates and the system of coordinates with a nonstandard synchronization, the "everyday," radio ("r"), synchronization, are exposed and exploited throughout the paper. For simplicity and for easier understanding, we shall mainly deal with the standard basis, but remembering that the approach with 4D geometric quantities holds for any choice of basis in M^4.

3. *The comparison of the usual transformations and the LT*

One should first make a remark regarding the figures in Ref. [1]. Both figures should contain the time axes, too. Fig. 1(a) (Fig. 1(b) and Fig. 2) is the projection onto the hypersurface $t' = const.$ ($t = const.$); the distances are simultaneously determined in the S' (S) frame. The LT cannot transform the hypersurface $t' = const.$ into the hypersurface $t = const.$, since the LT mix the spatial and temporal components of any 4D quantity. Hence, due to the relativity of simultaneity, the distances that are simultaneously determined in the S' frame cannot be transformed by the LT into the distances simultaneously determined in the S frame. In Fig. 1(a), the coordinates of the events associated with charges q and Q, which are at rest in S', are $(ct'_q = 0, x'_q, y'_q, z'_q = 0)$ and $(ct'_Q = 0, x'_Q = 0, y'_Q = 0, z'_Q = 0)$; both events are on the same hypersurface $t' = 0$. The coordinates of *the same events* in the laboratory frame S, which are obtained by the LT, are $(ct_q = \gamma\beta x'_q, x_q = \gamma x'_q, y_q = y'_q, z_q = 0)$ and $(ct_Q = 0, x_Q = 0, y_Q = 0, z_Q = 0)$. Now these events *are not* on the same hypersurface $t = 0$. This explains why $t' = const$ cannot be transformed by the LT into the hypersurface $t = const.$ and why both figures would need to contain the time axes as well. Obviously, such figures could be possible for the Galilean transformations with absolute time, but for the LT they are meaningless.

In Sec. III of Ref. [1], Jackson discusses "Lorentz transformations of the angular momentum between frames." He starts with the usual covariant definition of the angular momentum tensor $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, Eq. (8) in Ref. [1]. Notice that the standard basis $\{\gamma_\mu\}$, i.e., Einstein's system of coordinates, is implicit in that definition. In Ref. [1] the vector $\mathbf{L}$ is called the angular momentum and its components L_i are identified with the "space-space" components of $M^{\mu\nu}$. However, a physical interpretation is not given for another vector $\mathbf{L}_t$. Its components $L_{t,i}$ are identified with the three "time-space" components of $M^{\mu\nu}$ (we denote Jackson's K_i with $L_{t,i}$, $\mathbf{K}$ with $\mathbf{L}_t$). The same identification is supposed to hold both in S', the rest frame of the charges q and Q,

$$L'_i = (1/2)\varepsilon_{ikl} M'^{kl}, \quad L'_{t,i} = M'^{0i}, \tag{1}$$

and in S, the laboratory frame, the same relations as (1) but with unprimed quantities. Thus, it is assumed in Ref. [1] that the components L_i and $L_{t,i}$ are the components of $M^{\mu\nu}$ and accordingly L_i and $L_{t,i}$ transform in the same way as the corresponding components of $M^{\mu\nu}$ transform under the LT. For example, M^{12} component, which is L_3 according to (1), transforms under the LT as $M^{12} = \gamma(M'^{12} + \beta M'^{02})$. This assumption then leads to the usual transformations of the components of $\mathbf{L}$ that are given by Eq. (11) in Ref. [1], which are

$$L_1 = L'_1, L_2 = \gamma(L'_2 - \beta L'_{t,3}), L_3 = \gamma(L'_3 + \beta L'_{t,2}). \tag{2}$$

Of course, the same procedure of the identifications in both frames S' and S would give the transformations for the components $L_{t,i}$,

$$L_{t,1} = L'_{t,1}, L_{t,2} = \gamma(L'_{t,2} + \beta L'_3), L_{t,3} = \gamma(L'_{t,3} - \beta L'_2). \tag{3}$$

Jackson [1] did not write these usual transformations for $L_{t,i}$ (3), since in Ref. [1], as already said, the physical interpretation is not given for $\mathbf{L}_t$.

The characteristic feature of these usual transformations (2) and (3) is that the components, e.g., L_i in S, are expressed by the mixture of components L'_i and $L'_{t,i}$ from S'. This causes that the components of the 3D angular momentum $\mathbf{L}$ do not vanish in the laboratory frame S, even if they do in S'. In the case considered in Ref. [1], L_3 is different from zero due to the contribution from $L'_{t,2}$, see Eq. (13) in Ref. [1]. The time rate of L_3 is calculated in Sec. III in Ref. [1] and it gives that the torque T_3 is different from zero in S although it vanishes in S'.

It is worth noting that the result for T_3 in Ref. [1] can be simply obtained in another way, i.e., by means of the usual transformations for the 3D torque. Namely, in the same way as discussed above (Eqs. (2) and (3)), we can determine the components T_i of the torque $\mathbf{T}$ ($\mathbf{T} = \mathrm{d}\mathbf{L}/\mathrm{d}\mathbf{t}$) and the components $T_{t,i}$ of another torque $\mathbf{T}_t$ ($\mathbf{T}_t = \mathrm{d}\mathbf{L}_t/\mathrm{d}t$) by the identification in both frames with the "space-space" and the "time-space" components, respectively, of the torque four-tensor $N^{\mu\nu}$, $T_i = (1/2)\varepsilon_{ikl}N^{kl}$, $T_{t,i} = T^{0i}$, which gives the transformations

$$\begin{aligned} T_1 &= T'_1,\ T_2 = \gamma(T'_2 - \beta T'_{t,3}),\ T_3 = \gamma(T'_3 + \beta T'_{t,2}), \\ T_{t,1} &= T'_{t,1},\ T_{t,2} = \gamma(T'_{t,2} + \beta T'_3),\ T_{t,3} = \gamma(T'_{t,3} - \beta T'_2). \end{aligned} \tag{4}$$

Again, the transformed components T_i are expressed by the mixture of components T'_k and $T'_{t,k}$. Hence, the components of "physical" torque $\mathbf{T}$ do not vanish in S, even if they do in S'. Observe that Jackson [1,7] and other authors consider that only the "space-space" parts, L_i and T_i, i.e. the 3D vectors $\mathbf{L}$ and $\mathbf{T}$, are physical quantities. (Actually, it is almost generally accepted that the covariant quantities, e.g., $M^{\mu\nu}$, $N^{\mu\nu}$ are only auxiliary mathematical quantities from which "physical" 3D quantities, $\mathbf{L}$, $\mathbf{T}$ are deduced.) For the problem considered in Ref. [1] all T'_k are zero but the component T_3 in S is $\neq 0$ due to the contribution from $T'_{t,2}$. It can be easily shown that T_3 obtained from (4) is equal to T_z, which is found in Secs. II and III in Ref. [1]. So, all three methods for the calculation of T_3 give the same result - the torque T_3 is different from zero in S although it vanishes in S', which means that the principle of relativity is violated when the 3D quantities and their transformations are used.

It should be noted that the expressions for **E** and **B**, Eqs. (3a) and (3b) in Ref. [1], are also obtained by means of the usual transformations for these 3D fields, Ref. [7] Eqs. (11.148) and (11.149). In Ref. [7], these transformations are derived in a completely analogous way as it is used for the derivation of (2), (3) and (4). Namely the components of **B** and **E** are identified with the "space-space" and the "time-space" components, respectively, of the electromagnetic field strength tensor $F^{\mu\nu}$. In S', this identification yields

$$B_i' = (1/2c)\varepsilon_{ikl}F'^{lk}, \quad E_i' = F'^{i0}, \tag{5}$$

and the same relations but with unprimed quantities are supposed to hold in S, see Ref. [7] Sec. 11.9. Since the components of **B** and **E** are considered to be the components of $F^{\mu\nu}$, they, B_i and E_i, transform in the same way as the corresponding components of $F^{\mu\nu}$ transform under the LT. (It is worth noting that Einstein's fundamental work [8] is the earliest reference on covariant electrodynamics and on the identification of components of $F^{\mu\nu}$ with the components of the 3D **E** and **B**.) In that way the usual transformations for the components of **B** and **E** are obtained

$$\begin{aligned} B_1 &= B_1',\ B_2 = \gamma(B_2' - \beta E_3'/c),\ B_3 = \gamma(B_3' + \beta E_2'/c), \\ E_1 &= E_1',\ E_2 = \gamma(E_2' + \beta c B_3'),\ E_3 = \gamma(E_3' - \beta c B_2'), \end{aligned} \tag{6}$$

see, e.g., Ref. [7] Eq. (11.148). (The usual transformations for the components of the 3D **B** and **E** (6) were derived by Lorentz and Poincaré [9], and independently by Einstein [5], and subsequently quoted and used in different problems in almost every textbook and paper on relativistic electrodynamics.) The main feature of the usual transformations (6) is that, e.g., the transformed components B_i are expressed by the mixture of components E_k' and B_k'. If in S' there is only an electric field, as in the example considered in Ref. [1], then, according to (6), the observers in S will "see" both the transformed electric field and the new magnetic field, as in Ref. [1] Eqs. (3a) and (3b).

The comparison of the identifications (5) and (1) (and the same for T_i, $T_{t,i}$) shows that the components L_i (T_i) correspond to $-B_i$ and $L_{t,i}$ ($T_{t,i}$) to $-E_i$. In all these identifications the components of the 3D vectors **L**, $\mathbf{L}_t$ (**T**, $\mathbf{T}_t$) and **B**, **E** are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric ε tensor too. The super- and subscripts are used only on the components of the 4D quantities.

In Refs. [10 − 13], several objections were raised on the derivation of the usual transformations for **B** and **E** (6) and it was shown that the transformations (6) differ from the LT of the 4D quantities that represent the electric and magnetic fields. The same result was obtained in Refs. [2] and [3] for the usual transformations of **L** and $\mathbf{L}_t$, (2) and (3), and **T** and $\mathbf{T}_t$, (4). Some of these objections will be quoted here.

(i) First, $F^{\mu\nu}$ ($M^{\mu\nu}$, $N^{\mu\nu}$) are only components (numbers) that are (implicitly) determined in the standard basis $\{\gamma_\mu\}$, i.e., in Einstein's system of coordinates. In another system of coordinates that is different than the Einstein system of coordinates, e.g., differing in the chosen synchronization, all above identifications are impossible and meaningless. Let us explain it in more detail. As already mentioned, in Ref. [6] a drastically different, nonstandard, "r" synchronization is also exposed and exploited throughout the paper. The "r" synchronization is commonly used in everyday life and not Einstein's synchronization. In the "r" synchronization there is an absolute simultaneity. As explained in Ref. [14]: "For if we turn on the radio and set our clock by the standard announcement "..."at the sound of the last tone, it will be 12 o'clock," then we have synchronized our clock with the studio clock according to the "r" synchronization. The components x_r^μ of the position 1-vector x in the $\{r_\mu\}$ basis (with the "r" synchronization) are connected with the components x^μ of the same x in the standard basis $\{\gamma_\mu\}$ by the following relations $x_r^0 = x^0 - x^1 - x^2 - x^3$, $x_r^i = x^i$, Ref. [6]. It is seen that the time component x_r^0 in the $\{r_\mu\}$ basis is expressed by the combination of the time x^0 and space x^i components from the standard basis $\{\gamma_\mu\}$. Observe that both bases are taken in the same reference frame. A similar result holds for the components $F_r^{\mu\nu}$ ($M_r^{\mu\nu}$, $N_r^{\mu\nu}$) in the $\{r_\mu\}$ basis. For example, Ref. [6], $F_r^{10} = F^{10} - F^{12} - F^{13}$, which shows that the "time-space" components in the $\{r_\mu\}$ basis are expressed by the mixture of the "time-space" components and the "space-space" components from the $\{\gamma_\mu\}$ basis. If it is supposed that the identification (5) holds in the $\{r_\mu\}$ basis as well, i.e., that $E_{1r} = F_r^{10}$, then it follows that $E_{1r} = E_1 + cB_3 - cB_2$. The electric field in the $\{r_\mu\}$ basis is the combination of the electric field and the magnetic field from the $\{\gamma_\mu\}$ basis. Thus only in the $\{\gamma_\mu\}$ basis it holds that $E_i = F^{i0}$ (5) and similarly $L_{t,i} = M^{0i}$ (1), $T_{t,i} = N^{0i}$, etc. This is the reason why we always put the quotation marks in the expressions "time-space" and "space-space." This consideration explains the above mentioned assertion that the identifications (5), (1) and those for T_i, $T_{t,i}$ are impossible for some nonstandard synchronization, i.e., that the components are numbers that depend on the chosen system of coordinates.

(ii) However, components tell only part of the story, while the basis contains the rest of the information about the considered physical quantity. In the 4D spacetime, the physical quantity is not $F^{\mu\nu}$, since these components depend, e.g., on the chosen synchronization, but the 4D geometric quantity, $F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, which is the same quantity for all relatively moving inertial frames and for any chosen system of coordinates, see Eq. (8) and the discussion before Eq. (20). This fact is completely overlooked in all usual covariant approaches and in the above identifications and transformations. They deal only with components, which are implicitly determined in the standard basis. The identification (5) completely ignores the existence of the bivector basis $\gamma_\mu \wedge \gamma_\nu$. In all usual approaches, the 3D vectors, e.g., $\mathbf{B}'$ and $\mathbf{E}'$ in S', and $\mathbf{B}$ and $\mathbf{E}$ in S, which are geometric quantities in the 3D space, are constructed multiplying six independent components of $F'^{\mu\nu}$ and $F^{\mu\nu}$ (defined in the 4D spacetime) by the unit 3D vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ in S' and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in S, as in, e.g., Eq. (11.149) in Ref. [7],

$$\mathbf{E} = \gamma(\mathbf{E}' - \beta \times \mathbf{B}') + (\gamma^{\mathbf{2}}/\gamma + \mathbf{1})\beta(\beta \cdot \mathbf{E}). \tag{7}$$

The components, e.g., E_i in S, are determined by the usual transformations (6) from the components E_i' and B_i' in S', but there is no transformation which transforms $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ from S' into $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in S. Hence it is not true that $\mathbf{E} = \mathbf{E_1 i} + \mathbf{E_2 j} + \mathbf{E_3 k}$ is obtained by the LT from $\mathbf{E}'{=}\mathbf{E_1' i'} + \mathbf{E_2' j'} + \mathbf{E_3' k'}$ and $\mathbf{B}'$. The bivector basis $\gamma_\mu \wedge \gamma_\nu$ in the 4D spacetime is quite different than the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in the 3D space and there is no mathematically correct procedure for their identification. If, regardless of it, one assumes that in S', e.g., $E_2'\mathbf{j}'$ is identified with $F'^{20}\gamma_2' \wedge \gamma_0'$, then it is not possible to identify $E_2\mathbf{j}$ with $F^{20}\gamma_2 \wedge \gamma_0$ in S, since the term $\gamma_2' \wedge \gamma_0'$ does not transform into $\gamma_2 \wedge \gamma_0$; $\gamma_2 \wedge \gamma_0 = \gamma(\gamma_2' \wedge \gamma_0' - \beta\gamma_2' \wedge \gamma_1')$.

On the other hand, 1-vectors of the electric field E and the magnetic field B are defined in a mathematically correct way in terms of the electromagnetic field F (bivector) by the relations (12); all 4D quantities in (12) are AQs, i.e., they are defined without reference frames. When some basis is introduced then, e.g., E is the same quantity for all relatively moving inertial frames and for any chosen system of coordinates

$$E = E^\mu \gamma_\mu = E_r^\mu r_\mu = E'^\mu \gamma_\mu' = E_r'^\mu r_\mu', \tag{8}$$

where all primed quantities are the Lorentz transforms of the unprimed ones. The components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. (The LT in the $\{r_\mu\}$ basis are given in Ref. [6].) If the standard basis is chosen, then the components E^μ of E transform under the LT as the components of any 1-vector transform, as in (22),

$$E^0 = \gamma(E'^0 + \beta E'^1),\ E^1 = \gamma(E'^1 + \beta E'^0),\ E^{2,3} = E'^{2,3}. \tag{9}$$

These transformations are relativistically correct LT and, according to them, the components E'^μ transform by the LT again to E^μ, *there is no mixing of components.* Then, in contrast to the usual construction of the 3D vectors, the components E'^μ and E^μ are multiplied by the unit 1-vectors γ_μ' and γ_μ, respectively, and both the components and the basis 1-vectors in S are obtained by the LT from the corresponding quantities in S'. *The electric field* $E'^\mu\gamma_\mu'$ from S' *transforms by the LT again to the electric field* $E^\mu\gamma_\mu$ in S. Moreover, these LT refer to the same 4D quantity, as in (8). This is not the case with the usual transformations (6) and (7).

The transformations that do not refer to the same 4D quantity are not the LT, but we call them the "apparent" transformations (AT). According to that, the transformations (4) for $\mathbf{T}$, (2) for $\mathbf{L}$ (Eq. (11) in Ref. [1]) and (3) for $\mathbf{L}_t$, are not the LT, but they are the AT of the 3D $\mathbf{T}$ and $\mathbf{L}$. In Refs. [10 − 12] it is explained in detail that the usual transformations of $\mathbf{B}$ and $\mathbf{E}$, (6) and (7), or [7] Eqs. (11.148) and (11.149), are the AT and not the LT (a fundamental achievement). In fact, as shown in Ref. [2], all transformations of the 3D vectors $\mathbf{p}$, $\mathbf{F}$, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{L}$, $\mathbf{T}$, etc. are the AT and not the LT. This also means that equations with them, like Eq. (4) ($\mathrm{d}\mathbf{p}/\mathrm{d}\mathbf{t} = \mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$) and Eq. (5) ($\mathrm{d}\mathbf{L}/\mathrm{d}\mathbf{t} = \mathbf{T}$) in Ref. [1], are not relativistically correct equations. Thus, the fact that in Ref. [1] the same result, $\mathbf{T}' = \mathbf{0}$ but $\mathbf{T} \neq \mathbf{0}$, is obtained by two different methods of calculation does not mean that the calculations are relativistically correct and that the principle of relativity is satisfied.

4. Definitions of 4D quantities

The relativistically correct treatment of the problem considered in Ref. [1] and the resolution of Jackson's paradox are given in Ref. [2], Secs. 4-4.3, using 4D torques N, N_s and N_t. Here we shall only briefly expose the consideration from Secs. 2, 4, 4.1 and 4.3 of Ref. [2].

In almost all usual treatments of the electromagnetism, both in the tensor formalism and in the geometric algebra formalism, the theory of the electromagnetism is presented assuming that the components $F^{\mu\nu}$ do not have an independent existence but are defined either by the components of the 3D $\mathbf{E}$ and $\mathbf{B}$, Eq. (5), or by the components of the electromagnetic potential A, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In Ref. [15], an axiomatic formulation of the classical electromagnetism is presented that deals with the electromagnetic field F (bivector) as the primary quantity for the whole electromagnetism. Only the field equation with F is postulated

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c. \tag{10}$$

The source of the field is the electromagnetic current j, which is a 1-vector field and ∂ is the gradient operator (a 1-vector field). (A single field equation for F was first given by M. Riesz [16].) The F field yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors E and B or the potential A. For the given sources, the Clifford algebra formalism enables one to find in a simple way the electromagnetic field F. Namely the gradient operator ∂ is invertible and (10) can be solved giving that $F = \partial^{-1}(j/\varepsilon_0 c)$. (For more detail see Ref. [15].) In that way we can determine $F(x)$ for a charge Q moving with constant velocity u_Q (1-vector), which is

$$F(x) = kQ(x \wedge (u_Q/c))/\left| x \wedge (u_Q/c)\right|^3, \tag{11}$$

where $k = 1/4\pi\varepsilon_0$. For the charge Q at rest, $u_Q/c = \gamma_0$. Instead of the usual identification of E_i and B_i with components of $F^{\mu\nu}$, (5), we construct in a mathematically correct way, i.e., by a decomposition of F, the 4D geometric quantities that represent the electric and magnetic fields. F can be decomposed into 1-vectors of the electric field E and the magnetic field B and a unit time-like 1-vector v/c as

$$\begin{aligned} F &= (1/c)E \wedge v + (IB) \cdot v, \\ E &= (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v), \end{aligned} \tag{12}$$

see [11]. In (12) I is the unit pseudoscalar. (I is defined algebraically without introducing any reference frame, as in Ref. [17], Sec. 1.2.) v is the velocity (1-vector) of a family of observers which measures E and B fields. Since F is antisymmetric, it holds that $E \cdot v = B \cdot v = 0$, which yields that only three components of E and three components of B are independent quantities. However it does not mean that three spatial components of E, or B, are necesarilly independent components. Namely E and B depend not only on F but on v as well. Hence, the form of v in a given inertial frame will determine which three components are independent.

From (12) and the known F, (11), we find the 1-vectors E and B for a charge Q moving with constant velocity u_Q

$$\begin{aligned} E &= (D/c^2)[(x \wedge u_Q) \cdot v], \\ B &= (-D/c^3)I(x \wedge u_Q \wedge v), \end{aligned} \tag{13}$$

where $D = kQ/\left|x \wedge (u_Q/c)\right|^3$. Note that B in (13) can be expressed in terms of E as

$$B = (1/c^3)I(u_Q \wedge E \wedge v). \tag{14}$$

When the world lines of the observer and the charge Q coincide, $u_Q = v$, then (13) yields that $B = 0$ and only an electric field (Coulomb field) remains. E and B from (13) are AQs. They naturally generalize the expressions for the 3D **E** and **B**, Eqs. (3a) and (3b) in Ref. [1], to the 4D spacetime.

Furthermore, instead of the covariant form of the Lorentz force, Eq. (A2) in Ref. [1] (only components in the implicit $\{\gamma_\mu\}$ basis), we deal with the Lorentz force as an AQ,

$$K_L = (q/c)F \cdot u. \tag{15}$$

Using the decomposition of F into E and B, (12), this Lorentz force can be written as

$$K_L = (q/c)\left[(1/c)E \wedge v + (IB) \cdot v\right] \cdot u, \tag{16}$$

where u is the velocity (1-vector) of a charge q. Particularly, from the definition of the Lorentz force $K_L = (q/c)F \cdot u$ and the relation $E = (1/c)F \cdot v$ it follows that the Lorentz force seen by an observer comoving with a charge, $u = v$, is purely electric $K_L = qE$. Hence our equation of motion of a charge q is

$$m\,\mathrm{d}u/\mathrm{d}\tau = (q/c)\left[(1/c)E \wedge v + (IB) \cdot v\right] \cdot u, \tag{17}$$

which replaces (4) from Ref. [1]. In that equation τ is the proper time of the particle, u is the velocity 1-vector of the particle that is defined to be the tangent to its world line.

The 4D AQs, the angular momentum M and the torque N (bivectors) for the Lorentz force K_L and manifestly Lorentz invariant equation connecting M and N are defined as

$$M = x \wedge p, \quad N = x \wedge K_L; \quad N = \mathrm{d}M/\mathrm{d}\tau, \tag{18}$$

where x is the position 1-vector and p is the proper momentum (1-vector) $p = mu$.

When M and N are written as CBGQs in the $\{\gamma_\mu\}$ basis, they become

$$\begin{aligned} M &= (1/2)M^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \quad M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \\ N &= (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \quad N^{\mu\nu} = x^\mu K_L^\nu - x^\nu K_L^\mu. \end{aligned} \tag{19}$$

The components, e.g., $N^{\mu\nu}$, are determined as $N^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot N) = (\gamma^\nu \wedge \gamma^\mu) \cdot N$. We see that the components $M^{\mu\nu}$ are identical to the covariant angular momentum four-tensor given by (A3) in Ref. [1]. However M and N from (18) are 4D geometric quantities, the 4D AQs, whereas the components $M^{\mu\nu}$ and $N^{\mu\nu}$ that are used in the usual covariant approach, e.g., (A3) in Ref. [1], are coordinate quantities, the numbers obtained in the specific system of coordinates, i.e., in the $\{\gamma_\mu\}$ basis. In contrast to the usual covariant approach, M and N as 4D CBGQs are also 4D geometric quantities, which contain both components and a *basis*, here bivector basis $\gamma_\mu \wedge \gamma_\nu$. The essential difference between our geometric approach and the usual covariant picture is the presence of the basis. *The existence of a basis implies that every 4D CBGQ is invariant under the passive LT*; the components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. This means that a CBGQ represents *the same physical quantity* for relatively moving 4D observers. Hence it holds that, e.g.,

$$N = (1/2)N'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \tag{20}$$

where all primed quantities are the Lorentz transforms of the unprimed ones, see also (8). When physical laws are written with such Lorentz invariant quantities, 4D AQs or 4D CBGQs, then they automatically satisfy the principle of relativity. In the standard approach to special relativity [5], the principle of relativity is postulated outside the framework of a mathematical formulation of the theory. There, Ref. [5], it is also considered that the principle of relativity holds for the equations written with the 3D quantities.

Here, as in Ref. [2], we introduce new 4D torques as AQs, 1-vectors N_s and N_t. The same decomposition can be made for N as for F, (12). N is decomposed into two 1-vectors, the "space-space" torque N_s and the "time-space" torque N_t, and the unit time-like 1-vector v/c as

$$\begin{aligned} N &= (v/c) \cdot (IN_s) + (v/c) \wedge N_t, \\ N_s &= I(N \wedge v/c),\ N_t = (v/c) \cdot N;\ N_s \cdot v = N_t \cdot v = 0. \end{aligned} \tag{21}$$

Only three components of N_s and three components of N_t are independent since N is antisymmetric. Here again v is the velocity (1-vector) of an observer who measures N_s and N_t. Again, as for E and B, the torques N_s and N_t depend not only on the bivector N but on v as well. The relations (21) show that N_s and N_t taken *together* contain the same physical information as the bivector N.

When N_s and N_t are written as CBGQs in the $\{\gamma_\mu\}$ basis, they are

$$N_s = N_s^\mu \gamma_\mu = (1/2c)\varepsilon^{\alpha\beta\mu\nu}N_{\alpha\beta}v_\mu\gamma_\nu,\ N_t = (1/c)N^{\mu\nu}v_\mu\gamma_\nu.$$

In the frame of "fiducial" observers, in which the observers who measure 1-vectors E, B, K_L, N_s and N_t are at rest, the velocity v is $v = c\gamma_0$. In that frame with the $\{\gamma_\mu\}$ basis (it will be called the γ_0-frame) $v^\mu = (c, 0, 0, 0)$. Let us now take that the S frame is the γ_0-frame. Then in S, $N_s^0 = N_t^0 = 0$, and only the spatial components remain. N_s^i components are $N_s^1 = N^{23} = x^2 K_L^3 - x^3 K_L^2$, $N_s^2 = N^{31}$ and $N_s^3 = N^{12}$. Comparison with the identification $T_i = (1/2)\varepsilon_{ikl} N^{kl}$ shows that in the γ_0-frame the components of the 1-vector N_s correspond to components of "physical" 3D torque $\mathbf{T}$, $T_i = N_s^i$, and similarly $T_{t,i} = N_t^i$. Hence, only for "fiducial" observers and the $\{\gamma_\mu\}$ basis, i.e., in the γ_0-frame, one can deal with *components* of *two* 3D torques $\mathbf{T}$ and $\mathbf{T}_t$; all *six* components are equally well physical. However, even in that frame the relativistically correct *geometric* quantities are not 3D vectors $\mathbf{T}$ and $\mathbf{T}_t$, but 1-vectors N_s and N_t. The whole discussion with the torque can be completely repeated for the angular momentum replacing N, N_s and N_t by M, M_s and M_t. In the γ_0-frame components of 1-vectors M_s and M_t correspond to components of $\mathbf{L}$ and $\mathbf{L}_t$ respectively in the usual 3D picture.

Of course, it holds that, e.g., N_s as a CBGQ is a Lorentz invariant quantity, $N_s = N_s^\mu \gamma_\mu = N_s'^\mu \gamma_\mu'$, where again all primed quantities are the Lorentz transforms of the unprimed ones. The components of N_s (N_t, M_s, M_t, E, B, ..) transform under the LT as the components of any 1-vector transform

$$N_s'^0 = \gamma(N_s^0 - \beta N_s^1),\ N_s'^1 = \gamma(N_s^1 - \beta N_s^0),\ N_s'^{2,3} = N_s^{2,3}. \tag{22}$$

The LT of N_t^μ, M_s^μ, M_t^μ and of E^μ, B^μ, (9), are of the same form. In contrast to all mentioned AT for the components of the 3D vectors, e.g., (4), (2) and (3), and (6), the components N_s^μ transform again to $N_s'^\mu$; *there is no mixing of components*, as in (9).

Furthermore, we write N from (18) using the expression for K_L and equation (11) for F as

$$N = (Dq/c^2)(u \cdot x)(u_Q \wedge x). \tag{23}$$

N_s and N_t are then determined from (21) and (23)

$$\begin{aligned} N_s &= (Dq/c^3)(u \cdot x) I (x \wedge v \wedge u_Q), \\ N_t &= (Dq/c^3)(u \cdot x)[(x \wedge u_Q) \cdot v]. \end{aligned} \tag{24}$$

Comparison with (13) shows that N_s and N_t can be expressed in terms of B and E as

$$\begin{aligned} N_s &= q(u \cdot x) B, \\ N_t &= (q/c)(u \cdot x) E. \end{aligned} \tag{25}$$

As already said, in connection with (13), when $u_Q = v$ then $B = 0$ and $N_s = 0$ as well.

5. *Resolution of Jackson's paradox using 4D torques*

The knowledge of N as an AQ, (23), enables us to find the expressions for N as CBGQs in S' and S. First let us write all AQs from (23) as CBGQs in S', the rest frame of the charge Q, in which $u_Q = c\gamma'_0$. Then $N = (Dq/c)(u \cdot x)(\gamma'_0 \wedge x)$, and in the $\{\gamma'_\mu\}$ basis it is explicitly given as

$$\begin{aligned} N &= (1/2)N'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu = N'^{01}(\gamma'_0 \wedge \gamma'_1) + N'^{02}(\gamma'_0 \wedge \gamma'_2), \\ N'^{01} &= (Dq/c)(u'^\mu x'_\mu)x'^1,\ N'^{02} = (Dq/c)(u'^\mu x'_\mu)x'^2. \end{aligned} \tag{26}$$

The components x'^μ are $x'^\mu = (x'^0 = ct', x'^1, x'^2, 0)$ where $x'^1 = r'\cos\theta'$, $x'^2 = r'\sin\theta'$. In S', $u = u'^\mu\gamma'_\mu$, where $u'^\mu = dx'^\mu/d\tau = (u'^0, u'^1, u'^2, 0)$. The components $N'^{\mu\nu}$ that are different from zero are only N'^{01} and N'^{02}.

In order to find the torque N in S, we now write all AQs from (23) as CBGQs in S and in the $\{\gamma_\mu\}$ basis. In S the charge Q is moving with velocity $u_Q = \gamma_Q c\gamma_0 + \gamma_Q\beta_Q c\gamma_1$, where $\beta_Q = |\mathbf{u_Q}|/c$ and $\gamma_Q = (1-\beta_Q^2)^{-1/2}$. Then N is

$$N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = N^{01}\gamma_0 \wedge \gamma_1 + N^{02}\gamma_0 \wedge \gamma_2 + N^{12}\gamma_1 \wedge \gamma_2, \tag{27}$$

or explicitly it becomes

$$\begin{aligned} N &= (Dq/c)(u^\mu x_\mu)[\gamma_Q(x^1 - \beta_Q x^0)(\gamma_0 \wedge \gamma_1) \\ &\quad + \gamma_Q x^2(\gamma_0 \wedge \gamma_2) + \beta_Q\gamma_Q x^2(\gamma_1 \wedge \gamma_2)]. \end{aligned} \tag{28}$$

Now the components $N^{\mu\nu}$ that are different from zero are not only the "time-space" components N^{01} and N^{02} but also the "space-space" component $N^{12} = \beta_Q N^{02}$.

We note that another way to find N as CBGQ in S, (28), is to make the LT of N as CBGQ in S', (26). Of course, according to (20), both CBGQs are equal, N ((26)) $= N$ ((28)); they represent the same 4D quantity N from (23) in S' and S frames. The principle of relativity is naturally satisfied and there is no paradox.

Instead of the torque N we can use N_s and N_t. However, as already said, N_s and N_t are not uniquely determined by N, but their explicit values depend also on v. This means that it is important to know which frame is chosen to be the γ_0-frame. Observe that the same conclusions also refer to the determination of E, B and M_s, M_t from F and M, respectively. In this paper, we shall only consider the case when S is the γ_0-frame. For other cases readers can consult Sec. 4.2 in Ref. [2]. N_s and N_t will be determined directly from (24) taking into account that $v = c\gamma_0$ and $u_Q = \gamma_Q c\gamma_0 + \gamma_Q\beta_Q c\gamma_1$. This yields that

$$N_s = N_s^\mu\gamma_\mu = N^{12}\gamma_3,\ N_t = N_t^1\gamma_1 + N_t^2\gamma_2 = N^{01}\gamma_1 + N^{02}\gamma_2, \tag{29}$$

where N^{12}, N^{01} and N^{02} are from (28). Thus when S is the γ_0-frame *the "space-space" torque N_s is different from zero.*

The same N_s and N_t can be determined using (25) and (13). The charge Q moves in S, which yields that both E and *the magnetic field B are different from zero.* Then $E = E^\mu\gamma_\mu$, $E^0 = E^3 = 0$, $E^1 = D\gamma_Q(x^1 - \beta_Q x^0)$, $E^2 = D\gamma_Q x^2$, and the magnetic field is $B = B^\mu\gamma_\mu$, $B^0 = B^1 = B^2 = 0$, $B^3 = (D/c)\gamma_Q\beta_Q x^2 = \beta_Q E^2/c$.

The spatial components E^i and B^i are the same as the usual expressions for the components of **E** and **B** for an uniformly moving charge. Inserting these equations into (25) we again find N_s and N_t as in (29). N_s is $\neq 0$ since B is $\neq 0$.

N_s from (29) can be written in the form similar to (7) from Ref. [1], when the above explicit forms of E and B are used. Thus

$$N_s = N^3\gamma_3 = N^{12}\gamma_3 = (\beta_Q ctK_L^2 + (q/c)\beta_Q y(E^\mu u_\mu))\gamma_3. \tag{30}$$

In the usual approach, e.g., Ref. [1], it is considered that in the S frame the whole physical torque is the 3D **T**, i.e., T_z, given by (7) in Ref. [1]. We see that in the 4D spacetime, the physical torque, theoretically and *experimentally,* is either the bivector N, (28) or (26), or *two* 1-vectors N_s and N_t given by (29) and (30). Only when the laboratory frame S is the γ_0-frame, the spatial components of N_s can be put into the correspondence with the components of the 3D **T**. However note that in (30) all components are the components of the 4D quantities, 1-vectors x, u, K_L, E and B, while in (7) in Ref. [1] only the corresponding 3D vectors are involved.

Let us now determine N_s and N_t as CBGQs in S'. Relative to the S' frame, the charge Q is at rest $u_Q = c\gamma_0'$, but the "fiducial" observers are moving with velocity $v = \gamma_Q c\gamma_0' - \gamma_Q\beta_Q c\gamma_1'$. Then N_s and N_t in S' can be obtained either directly from (24) or by means of the LT (22) of N_s and N_t from (29). We find N_s and N_t as

$$\begin{aligned} N_s &= N_s'^{\mu}\gamma_\mu' = N_s'^{3}\gamma_3',\ N_s'^{3} = \gamma_Q\beta_Q N'^{02}, \\ N_t &= N_t'^{\mu}\gamma_\mu',\ N_t'^{0} = -\beta_Q\gamma_Q N'^{01},\ N_t'^{1,2} = \gamma_Q N'^{01,2},\ N_t'^{3} = 0, \end{aligned} \tag{31}$$

where N'^{01}, N'^{02} are given in (26). Now, N_s is different from zero not only in S, but in the S' frame as well. The same results for N_s and N_t in S' can be obtained using (25) and (13) and writing all AQs as CBGQs in the S' frame.

It can be easily seen that N_s (N_t) from (29) is equal to N_s (N_t) from (31); it is the same 4D CBGQ for observers in S and S'; the principle of relativity is naturally satisfied and there is no paradox.

Inserting N_s and N_t from (29) and (31) into (21), which connects N with N_s and N_t, we find that the expressions for N in S and S' are the same, as it must be.

If we would take that S' is the γ_0-frame, as in Sec. 4.2 in Ref. [2], then the explicit expressions for N_s and N_t as CBGQs would be different than those given in (29) and (31). For example, in that case N_s is zero, but $N_t \neq 0$ both in S' and S. However, when these new expressions for N_s and N_t as CBGQs are inserted into (21), they will give the same N as when S is the γ_0-frame.

6. Conclusions

It is proved in this paper, as in Ref. [2], that the physical torques are the 4D geometric quantities, the bivector N defined in (18) and (23), or the 1-vectors N_s and N_t that are derived from N according to (21). They together contain the same physical information as the bivector N. In the considered case N_s and N_t are defined in (24). Only in the γ_0-frame one can deal with components T_i and $T_{t,i}$ of two 3D torques $\mathbf{T}$ and $\mathbf{T}_t$, respectively. In that frame, the temporal components of N_s and N_t as CBGQs are zero, $N_s^0 = N_t^0 = 0$, and the spatial components are $T_i = N_s^i$ and $T_{t,i} = N_t^i$. All components T_i and $T_{t,i}$ are equally well physical for "fiducial" observers. Hence, it is not true, as generally accepted, that only the 3D torque $\mathbf{T}$ is a well-defined physical quantity. However, it is shown here, and in Ref. [2], that even in the frame of "fiducial" observers, the relativistically correct geometric quantities are not 3D vectors $\mathbf{T}$ and $\mathbf{T}_t$, but the bivector N or 1-vectors N_s and N_t. These 4D geometric quantities correctly transform under the LT, e.g., the whole torque N_s remains unchanged under the passive LT, $N_s = N_s^\mu \gamma_\mu = N_s'^\mu \gamma_\mu'$, where the components transform by means of the LT (22) and the basis 1-vectors γ_μ by the inverse LT. This means that the principle of relativity is satisfied and the paradox with the torque does not appear. In contrast to it, the components T_i and $T_{t,i}$ transform according to the AT (4), which differ from the LT for components of 1-vectors, e.g., (22). Furthermore, the objections (i) and (ii) from Sec. 2 show that the transformations of $\mathbf{T}$ and $\mathbf{T}_t$, as geometric quantities in the 3D space, are not the LT but the relativistically incorrect AT.

The validity of the above relations with 4D geometric quantities can be experimentally checked measuring all six independent components of N, or N_s and N_t taken together, in both relatively moving frames. Only such complete data are physically relevant in the 4D spacetime. Remember that the usual 3D torque $\mathbf{T}$ is connected only with three spatial components of N_s in the frame of "fiducial" observers. These three components are not enough for the determination of the relativistically correct 4D torques N, or N_s and N_t. This is the real cause of Jackson's paradox.

References

[1] J. D. Jackson, Am. J. Phys. **72** (2004) 1484.

[2] T. Ivezić, Found. Phys. **36** (2006) 1511.

[3] T. Ivezić, Found. Phys. **37** (2007) 747.

[4] D. Hestenes, Am. J. Phys. **71 (**2003) 691; *Space-Time Algebra*, Gordon & Breach, New York (1966); *New Foundations for Classical Mechanics*, Kluwer, Dordrecht (1999) 2nd ed.; C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge (2003).

[5] A. Einstein, Ann. Physik. (Leipzig) **17** (1905) 891; transl. by W. Perrett and G. B. Jeffery, in *The Principle of Relativity*, Dover, New York (1952).

[6] T. Ivezić, Found. Phys. **31** (2001) 1139.

[7] J. D. Jackson, *Classical Electrodynamics*, Wiley, New York (1998), 3rd ed.

[8] A. Einstein, Ann. Physik **49** (1916) 769, transl. by W. Perrett and G .B. Jeffery, in *The Principle of Relativity*, Dover, New York, (1952).

[9] H. A. Lorentz, Proc. Academy of Sciences of Amsterdam, **6** (1904), in W. Perrett and G.B. Jeffery, in *The Principle of Relativity*, Dover, New York, 1952; H. Poincaré, Rend. del Circ. Mat. di Palermo **21** (1906) 129. See also two fundamental Poincaré's papers

with notes by Logunov, A. A. Logunov, Hadronic J. **19** (1996) 109, in which it is shown that Poincaré was the first who gave a mathematically valid derivation of the standard transformations of the 3D **E** and **B**.

[10] T. Ivezić, Found. Phys. **33** (2003) 1339.

[11] T. Ivezić, Found. Phys. Lett. **18** (2005) 301.

[12] T. Ivezić, Found. Phys. **35** (2005) 1585.

[13] T. Ivezić, Phys. Rev. Lett. **98** (2007) 108901.

[14] C. Leubner, K. Aufinger and P. Krumm, Eur. J. Phys. **13** (1992) 170.

[15] T. Ivezić, Found. Phys. Lett. **18** (2005) 401.

[16] M. Riesz, *Clifford Numbers and Spinors*, Lecture Series No. 38, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland (1958).

[17] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Reidel, Dordrecht (1984).

JACKSONOV PARADOKS I NJEGOVO RJEŠENJE ČETIRIDIMENZIJSKIM GEOMETRIJSKIM VELIČINAMA

Pokazuje se kako je stvarni uzrok Jacksonovog paradoksa primjena tridimenzijskih (3D) veličina, npr. **E**, **B**, **F**, **L**, **T**, njihovih transformacija i relacija. Princip relativnosti je prirodno ispunjen i ne javljaju se paradoksi ako se fizička realnost pridjeli 4D geometrijskim veličinama, npr. 4D momentu sile N (bivektoru), ili, jednakovaljano, 4D momentima sile N_s i N_t (1-vektorima), koji zajedno sadrže istu fizikalnu informaciju kao i bivektor N.