NEW EXACT SOLUTIONS TO THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION

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Using the modified mapping method and the extended mapping method, we derive some new exact solutions of the generalized nonlinear Schrödinger equation, which are linear combinations of two different Jacobi elliptic functions and we also consider the solutions in the limiting cases.

1. Introduction

In order to study waves in nonlinear media, important objects are the travelling-wave solutions which describe the waves moving at a constant velocity. Particularly, we are interested in three types of travelling waves. The first are the solitary waves, which are localized travelling waves tending asymptotically to zero at large distances, the second are the periodic waves and the third are the kink waves, which rise or descend from one asymptotic state to another.

As for such solutions, there are many methods for finding special solutions to nonlinear partial differential equations. Some of the most fundamental methods are the Backlund transforms (see Refs. [1], [2] and [3]), the algebraic method (see Ref. [4]), the tanh method (see Refs. [5] and [6]), the balance method (see Ref. [7]), the Jacobi elliptic-functions method and its extensions (see Refs. [8], [9] and [10]), and so on.

As for the Jacobi elliptic-functions method for solutions of nonlinear partial differential equations, we are interested in exact periodic wave solutions. As is known, there are three basic Jacobi elliptic functions, $\operatorname{sn}\xi = \operatorname{sn}(\xi/m)$, $\operatorname{cn}\xi = \operatorname{cn}(\xi/m)$, and $\operatorname{dn}\xi = \operatorname{dn}(\xi/m)$, when $m = 1$ and m is the modulus of the elliptic functions. Jacobi elliptic functions satisfy the following relations:

$$\operatorname{sn}^2\xi + \operatorname{cn}^2\xi = 1, \quad \operatorname{dn}^2\xi + m^2\operatorname{sn}^2\xi = 1, \quad (\operatorname{sn}\xi)' = \operatorname{cn}\xi\,\operatorname{dn}\xi,$$

$$(\operatorname{cn}\xi)' = -\operatorname{sn}\xi\,\operatorname{dn}\xi, \quad (\operatorname{dn}\xi)' = -m^2\operatorname{sn}\xi\,\operatorname{cn}\xi.$$

Especially, when $m \to 1$, the Jacobi elliptic functions degenerate to the following functions:

$$\operatorname{sn}\xi \to \tanh\xi, \quad \operatorname{cn}\xi \to \operatorname{sech}\xi, \quad \operatorname{dn}\xi \to \operatorname{sech}\xi,$$

and when $m \to 0$, the Jacobi elliptic functions degenerate to the following functions:

$$\operatorname{sn}\xi \to \sin\xi, \quad \operatorname{cn}\xi \to \cos\xi, \quad \operatorname{dn}\xi \to 1.$$

Recently, the Jacobi elliptic function solutions of the nonlinear Schrödinger equations have been derived by the modified mapping method (see Ref. [11]). In Ref. [11], a mapping method and its extensions have been successfully used to obtain various Jacobi elliptic function solutions.

In this paper, we consider the following equation

$$iu_t + u_{xx} + \alpha|u|^2u + i\Big[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u^2|)_x u\Big] = 0. \tag{1}$$

This equation has important application in physics and we can obtain its exact solutions under some conditions.

2. Exact solutions to the generalized nonlinear Schrödinger equation

We seek travelling-wave solution to Eq. (1) in the following form,

$$u(x,t) = \phi(\xi)\exp(i(Kx - \Omega t)), \quad \xi = k(x - ct). \tag{2}$$

By virtue of (1) and (2), we get

$$i\Big(\gamma_1 k^3\phi''' - 3\gamma_1 K^2 k\phi' + \gamma_2 k\phi^2\phi' + 2\gamma_3 k\phi^2\phi' - ck\phi' + 2Kk\phi'\Big)$$

$$+\Big(\Omega\phi + k^2\phi'' - K^2\phi + \alpha\phi^3 + 3\gamma_1 Kk^2\phi'' + \gamma_1 K^3\phi - \gamma_2 K\phi^3\Big) = 0. \tag{3}$$

where γ_i $(i = 1, 2, 3)$, α and k are positive constants and the prime means differentiation with respect to ξ.

Then we have the two following equations

$$\gamma_1 k^2 \phi''' + \left(2K - c - 3\gamma_1 K^2\right)\phi' + \gamma_2 \phi^2 \phi' + 2\gamma_3 \phi^2 \phi' = 0 \,, \tag{4}$$

$$k^2 \left(1 - 3\gamma_1 K\right)\phi'' + \left(\Omega - K^2 + \gamma_1 K^3\right)\phi + \left(\alpha - \gamma_2 K\right)\phi^3 = 0 \,. \tag{5}$$

Integrating Eq. (4) and taking zero be the integration constant, we have

$$\gamma_1 k^2 \phi'' + (2K - c - 3\gamma_1 K^2)\phi + \left(\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3\right)\phi^3 = 0 \,. \tag{6}$$

Equations (5) and (6) have the same solution. So, we have the following equation

$$\frac{\gamma_1 k^2}{(1 - 3\gamma_1 K)k^2} = \frac{2K - c - 3\gamma_1 K^2}{\Omega - K^2 + \gamma_1 K^3} = \frac{\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3}{\alpha - \gamma_2 K} \,. \tag{7}$$

From Eq. (7), we obtain

$$K = \frac{C - \alpha\gamma_1}{3C\gamma_1 - \gamma_1\gamma_2}, \quad \Omega = \frac{(\alpha - \gamma_2 K)(2K - c - 3\gamma_1 K^2)}{C} + K^2 - \gamma_1 K^3 \,. \tag{8}$$

We assume that

$$A = \gamma_1 k^2, \qquad B = 2K - c - 3\gamma_1 K^2, \qquad C = \frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3 \,. \tag{9}$$

So, Eq. (6) is transformed into the following form,

$$A'' + B\phi + C\phi^3 = 0 \,. \tag{10}$$

According to the modified mapping method, we assume that Eq. (10) has the solution of the form

$$\phi(\xi), = A_0 + A_1 f + B_1 f^{-1} \,, \tag{11}$$

where A_i and B_i are constants to be determined and f satisfies the following equation,

$$f'^2 = pf^2 + \frac{1}{2}qf^4 + r \,, \tag{12}$$

where p, q and r are constants to be determined.

Due to Eqs. (10), (11) and (12), we get the following equations,

$$2rAB_1 + CB_1^3 = 0,$$

$$3CB_1^2 A_0 = 0,$$

$$pAB_1 + BB_1 + 3CB_1 A_0^2 + 3CB_1^2 A_1 = 0,$$

$$BA_0 + CA_0^3 + 6CB_1 A_0 A_1 = 0, \qquad (13)$$

$$pAA_1 + BA_1 + 3CA_0^2 A_1 + 3CB_1 A_1^2 = 0,$$

$$3CA_0 A_1 = 0,$$

$$qAA_1 + CA_1^3 = 0.$$

From Eqs. (13), we obtain

$$A_0 = 0, \quad A_1 = \pm\sqrt{\frac{-qA}{C}}, \quad B_1 = \pm\sqrt{\frac{-rA}{C}}, \quad pA + B + 3CA_1B_1 = 0. \quad (14)$$

From Eqs. (12), (8) and (14), we obtain the new exact solutions of Eq. (6),

$$u = \left[\pm\sqrt{\frac{-3q\gamma_1}{\gamma_2 + 2\gamma_3}}\, k f(\xi) \pm \sqrt{\frac{-6r\gamma_1}{\gamma_2 + 2\gamma_3}}\, k f^{-1}(\xi) \right] \exp(\mathrm{i}(Kx - \Omega t)), \quad (15)$$

where f satisfies Eq. (13). From Eqs. (8) and (15), we derive

$$\xi = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1(p \pm 3\sqrt{2qr}\,)}}\,(x - ct).$$

In the following, we discuss specific expressions for f given by Eq. (12) as example. There are three cases we consider.

Case 1: $p = 2$, $q = 2$, $r = 1$. Eq. (12) has solution $f(\xi) = \tanh\xi$. So, we have the new solitary-wave solution, which is also the solution of (1),

$$u = \left\{ \pm\tanh[k(x - ct)] \pm \coth[k(x - ct)] \right\} \sqrt{\frac{-6r\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \, \exp(\mathrm{i}(Kx - \Omega t)),$$

where $k = \sqrt{\dfrac{c + 3\gamma_1 K^2 - 2K}{2\gamma_1}}$ or $k = \sqrt{\dfrac{-(c + 3\gamma_1 K^2 - 2K)}{\gamma_1}}$.

Case 2: $p = 1 + m^2$, $q = 2m^2$, $r = 1$. Eq. (12) has the solution $f(\xi) = \mathrm{sn}\,\xi$ or $f(\xi) = \mathrm{cd}\,\xi$, and we obtain the periodic-wave solution of Eq. (1),

$$u = \left\{ \pm m\,\mathrm{sn}[k(x - ct)] \pm \mathrm{ns}[k(x - ct)] \right\} \sqrt{\frac{-6r\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \, \exp(\mathrm{i}(Kx - \Omega t)), \quad (16)$$

or

$$u = \left\{ \pm m\,\mathrm{cd}[k(x - ct)] \pm \mathrm{dc}[k(x - ct)] \right\} \sqrt{\frac{-6r\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \, \exp(\mathrm{i}(Kx - \Omega t)), \quad (17)$$

where

$$k = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1(1 + m^2 \pm 2m)}}\,,$$

and as $m \to 1$, Eq. (17) degenerates to Eq. (16).

Case 3: $p = 2 - m^2$, $q = -2$ and $r = -(1 - m^2)$. Eq. (12) has the solution $f(\xi) = \operatorname{dn}\xi$. The periodic-wave solution of Eq. (1) has the following form,

$$u = \left\{ \pm \operatorname{dn}[k(x - ct)] \pm \sqrt{1 - m^2}\,\operatorname{nd}[k(x - ct)] \right\} \sqrt{\frac{-6r\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \exp(\mathrm{i}(Kx - \Omega t))\,, \quad (18)$$

where

$$k = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1(2 - m^2 \pm 6\sqrt{1 - m^2})}}\,,$$

and as $m \to 1$, Eq. (18) degenerates to

$$u = \pm \operatorname{sech}[k(x - ct)] \sqrt{\frac{-6\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \exp(\mathrm{i}(Kx - \Omega t))\,,$$

where

$$k = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1}}\,.$$

Case 4: $p = 2 - m^2$, $q = 2(1 - m^2)$ and $r = 1$. From Eq. (12), we get the solution $f(\xi) = \operatorname{sc}\xi$. So, we obtain the solution of (1) in the following form,

$$u = \left\{ \pm \sqrt{1 - m^2}\,\operatorname{sc}[k(x - ct)] \pm \operatorname{cs}[k(x - ct)] \right\} \sqrt{\frac{-6\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \exp(\mathrm{i}(Kx - \Omega t))\,, \quad (19)$$

where

$$k = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1(2 - m^2 \pm 6\sqrt{1 - m^2})}}\,,$$

and as $m \to 1$, Eq. (19) degenerates to

$$u = \pm \operatorname{cs}[k(x - ct)] \sqrt{\frac{-6\gamma_1}{\gamma_2 + 2\gamma_3}}\, k \exp(\mathrm{i}(Kx - \Omega t))\,,$$

where

$$k = \sqrt{\frac{c + 3\gamma_1 K^2 - 2K}{\gamma_1}}\,.$$

Furthermore, using the extended mapping method, we obtain new exact solutions of Eq. (1). We assume that Eq. (11) has the solution of the form,

$$\phi(\xi) = A_0 + A_1 f + B_1 g \,, \tag{20}$$

where A_i and B_i are constants to be determined and f and g satisfy the following equations,

$$f'^2 = pf^2 + \frac{1}{2}qf^4 + r \,, \tag{21}$$
$$g'' = g(c_1 + c_2 f^2) \,, \quad g^2 = c_3 + c_4 f^2 \,.$$

By [12], we have the following equations:

$$
\begin{aligned}
f^0 : \quad & BA_0 + C(A_0^3 + 3c_3 A_0 B_1^2) = 0 \,, \\
g : \quad & c_1 AB_1 + BB_1 + C(3A_0^2 B_1 + c_3 B_1^3) = 0 \,, \\
f : \quad & pAA_1 + BA_1 + C(3A_0^2 A_1 + 3c_3 A_1 B_1^2) = 0 \,, \\
fg : \quad & 6CA_0 A_1 B_1 = 0 \,, \\
f^2 : \quad & C(3A_0 A_1^2 + 3c_4 A_0 B_1^2) = 0 \,, \\
f^2 g : \quad & c_2 AB_1 + 3CA_1^2 B_1 + c_4 CB_1^3 = 0 \,,
\end{aligned} \tag{22}
$$

From Eq. (21), we find that

$$
\begin{aligned}
& A_0 = 0 \,, \; A_1 = \pm \sqrt{\frac{(c_4 p - c_3 q)A + c_4 B}{c_3 C}} \,, \\
& B_1 = \pm \sqrt{\frac{-(pA + B)}{3c_3 C}} \,, \\
& (3c_4 p - 3c_3 q - c_1 c_4 + c_2 c_3)A + 2c_4 B = 0 \,, \\
& (3c_1 - p)A + 2B = 0 \,.
\end{aligned} \tag{23}
$$

By Eqs. (20) and (22), we obtain the new exact solution of Eq. (1) and it reads

$$
u = \left[\pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)(3c_1 c_4 + 2c_3 q - 3c_4 p)}{(\gamma_2 + 2\gamma_3)c_3(3c_1 - p)}} \, f(\xi) \right.
$$

$$
\left. \pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)c_1}{(\gamma_2 + 2\gamma_3)c_3(3c_1 - p)}} \, g(\xi) \right] \exp(\mathrm{i}(Kx - \Omega t)) \,,
$$

where f and g satisfy Eq. (20) and

$$
\xi = \sqrt{\frac{-2(2K - c - 3\gamma_1 K^2)}{\gamma_1(3c_1 - p)}} \, (x - ct) \,.
$$

Next, we discuss the specific expressions for f and g according to Eq. (20) as examples, and there are three cases we consider.

Case 1: $p = -(1 - m^2)$, $q = 2$, $r = m^2$.

(i): $c_1 = -m^2$, $c_2 = 2$, $c_3 = -1$, $c_4 = 1$; Eq. (20) has the solutions $f(\xi) = \text{ns } \xi$, $g(\xi) = \text{cs } \xi$. So, we have the new periodic wave solution of Eq. (1),

$$u = \left[\pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)(1 - 2m^2)}} \; \text{ns}[k(x - ct)] \right.$$

$$\left. \pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)m^2}{(\gamma_2 + 2\gamma_3)(1 - 2m^2)}} \; g(\xi) \right] \exp(\text{i}(Kx - \Omega t)), \tag{24}$$

where

$$k = \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)m^2}{(\gamma_2 + 2\gamma_3)(1 - 2m^2)}} \; .$$

As $m \to 0$ and $m \to 1$, Eq. (24) degenerates to

$$u = \pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)(1 - 2m^2)}} \; \text{ns}[k(x - ct)] \exp(\text{i}(Kx - \Omega t)),$$

where

$$k = \sqrt{\frac{-2(2K - c - 3\gamma_1 K^2)m^2}{\gamma_1}} \; ,$$

and

$$u = \pm \sqrt{\frac{-3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)}} \; \left\{ \text{ns}[k(x - ct)] + \text{cs}[k(x - ct)] \right\} \exp(\text{i}(Kx - \Omega t)),$$

where

$$k = \sqrt{\frac{2(2K - c - 3\gamma_1 K^2)m^2}{\gamma_1}} \; .$$

(ii): $c_1 = -1$, $c_2 = 2$, $c_3 = -m^2$, $c_4 = 1$. Eq. (20) has the solutions $f(\xi) = \text{ns } \xi$, $g(\xi) = \text{ds } \xi$. So, we have the new periodic wave solution of Eq. (1),

$$u = \left[\pm \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)(m^2 - 2)}} \; \text{ns}[k(x - ct)] \right.$$

$$\pm\sqrt{\frac{3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)m^2(m^2 - 2)}} \ \mathrm{ds}[k(x - ct)] \Bigg] \exp(\mathrm{i}(Kx - \Omega t)), \qquad (24)$$

where

$$k = \sqrt{\frac{3(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)(m^2 - 2)}} \, .$$

(iii): $c_1 = -1$, $c_2 = 2$, $c_3 = -m^2/(1 - m^2)$, $c_4 = 1/(1 - m^2)$, and

(iv): $c_1 = -m^2$, $c_2 = 2$, $c_3 = -1/(1 - m^2)$, $c_4 = 1/(1 - m^2)$,

are cases for which one can easily express the solutions of Eq. (1.1). So, we omit them here.

Case 2: $p = 2m^2 - 1$, $q = -2m^2$, $r = 1 - m^2$, $c_1 = m^2$, $c_2 = -2m^2$, $c_3 = 1 - m^2$, $c_4 = m^2$. In this case, we have $f(\xi) = \mathrm{cn} \ \xi$, $g(\xi) = \mathrm{dn} \ \xi$. So, we have the new periodic wave solution of Eq. (1),

$$u = \Bigg[\pm\sqrt{\frac{-3(2K - c - 3\gamma_1 K^2)m^2}{(\gamma_2 + 2\gamma_3)(m^2 + 1)}} \ \mathrm{cn}[k(x - ct)]$$

$$\pm\sqrt{\frac{3(2K - c - 3\gamma_1 K^2)m^2}{(\gamma_2 + 2\gamma_3)(1 - m^2)(m^2 + 1)}} \ \mathrm{dn}[k(x - ct)] \Bigg] \exp(\mathrm{i}(Kx - \Omega t)),$$

where

$$k = \sqrt{\frac{-2(2K - c - 3\gamma_1 K^2)}{(\gamma_2 + 2\gamma_3)(m^2 + 1)}} \, .$$

Case 3: $p = 2m^2 - 1$, $q = -2(m^2 - 1)$, $r = -m^2$, $c_1 = m^2$, $c_2 = 1 - 2m^2$, $c3 = -1$, $c_4 = 1$. In this case, we have $f(\xi) = \mathrm{nc} \ \xi$, $g(\xi) = \mathrm{sc} \ \xi$. So, we have the new periodic wave solution of Eq. (1),

$$u = \Bigg[\pm\sqrt{\frac{-3(2K - c - 3\gamma_1 K^2)(m^2 - 1)}{(\gamma_2 + 2\gamma_3)(m^2 + 1)}} \ \mathrm{nc}[k(x - ct)]$$

$$\pm\sqrt{\frac{-3(2K - c - 3\gamma_1 K^2)m^2}{(\gamma_2 + 2\gamma_3)(m^2 + 1)}} \ \mathrm{sc}[k(x - ct)] \Bigg] \exp(\mathrm{i}(Kx - \Omega t)), \qquad (25)$$

where

$$k = \sqrt{\frac{m^2(2K - c - 3\gamma_1 K^2)}{\gamma_1(m^2 + 1)}} \, .$$

As $m \to 1$, Eq. (23) degenerates to

$$u = \pm \frac{1}{2} \sqrt{\frac{-3(2K - c - 3\gamma_1 K^2)}{\gamma_2 + 2\gamma_3}} \ \mathrm{sc}[k(x - ct)] \exp(\mathrm{i}(Kx - \Omega t)) \,,$$

where

$$k = \sqrt{\frac{(2K - c - 3\gamma_1 K^2)}{2\gamma_1}} \,.$$

3. Conclusion

In summary, we obtained some new exact travelling-wave solutions of the generalized nonlinear Schrödinger equation via the Jacobi elliptic-functions method, which was used to find new exact travelling-wave solutions of nonlinear partial differential equations (NPDEs). As some special examples, these new exact travelling-wave solutions can degenerate into the kink-type solitary wave solutions. So, with the suggested Jacobi elliptic-functions method one can obtain easily the generalized soliton solutions, kink-type solitary solutions and travelling wave solutions for NDDEs.

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NOVA EGZAKTNA RJEŠENJA POOPĆENE NELINEARNE SCHRÖDINGEROVE JEDNADŽBE

Izvodimo neka nova egzaktna rješenja poopćene nelinearne Schrödingerove jednadžbe primjenom izmijenjene i proširene pridružne metode. Rješenja su linearni sastavci dviju Jacobijevih eliptičkih funkcija. Razmatramo također granična rješenja.