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HYPERBOLIC MOTION TREATMENT FOR BELL'S SPACESHIP EXPERIMENT

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In Bell's "spaceship" experiment, two spaceships that are initially at rest in some common inertial reference frame, are connected by a taut string. At time zero in the common inertial frame, both spaceships start accelerating, with a constant proper acceleration **a** as measured by an on-board accelerometer. Question: does the string break, i.e. does the distance between the two spaceships increase? We will present two treatments, one that uses only Minkowski spacetime diagrams and a second approach that uses the equations of accelerated motion in special relativity. The latter approach allows the calculation of the distance between rockets as well as the strain force in the string as a function of proper time. For simplicity, throughout the paper, all objects (string, rockets) are considered as being Born-rigid, thus neglecting the very minor effects on the length of the objects during the accelerated motion. The subject of the Bell paradox is encountered frequently in relativity graduate courses, but a complete, realistic solution has not been published to date.

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1. Analysis in current literature

In a simpler variant of Bell's thought experiment [1], both spaceships stop accelerating after a certain period of time previously agreed upon. The captain of each ship shuts off his engine after this time period has passed, as measured by an ideal clock carried on board of his ship. This allows comparisons before and after acceleration in suitable inertial reference frames in the sense of elementary special relativity. According to discussions by Dewan and Beran [2], in the spaceship launcher's reference system (ground frame), the distance between the ships will remain constant while the elastic limit of the string is length contracted, so that at a certain point in time the string should break. In this paragraph we will treat the spaceships as point masses and only consider the length of the string. This restriction will be removed in the next paragraph. We will analyze the variant case previously mentioned, where both spaceships shut of their engines after some time period T. The world lines (navy blue curves) of two observers A and B, who accelerate in the same direction with the same constant magnitude acceleration, are shown in Fig. 1. At A' and B', the observers stop accelerating. The dotted lines are "lines of simultaneity" for observer A. Is the space-like line segment **A'B"** longer than the space-like line segment **AB**? According to the discussions by Dewan and Beran [2], in the "spaceship-launcher's" reference frame S (i.e. in the ground frame), the distance L between the spaceships (A and B) must remain constant "*by definition*". Referring to the space-time diagram, we can see that both spaceships will stop accelerating at events A' and B', which are simultaneous in the launching frame S. From the previous argument, we can say that the length of the line segment **A'B'** equals the length of the line segment **AB**, which is equal to the initial distance L between spaceships before they started accelerating. We can also say that the velocities of A and B in frame S, after the end of the acceleration phase, are equal to v. Finally, we can say that the proper distance between spaceships A and B after the end of the acceleration phase in a co-moving frame is equal to the Lorentz length of the line segment **A'B"**. The line **A'B"** is defined to be a line of constant t', where t' is the time coordinate in the co-moving frame S', a time coordinate which can be computed from the coordinates in frame S via the Lorentz transform. Transformed into a frame co-moving with the spaceships, the line **A'B"** is a line of constant t' by definition and represents a line between the two ships "at the same time" as simultaneity is defined in the co-moving frame S'.

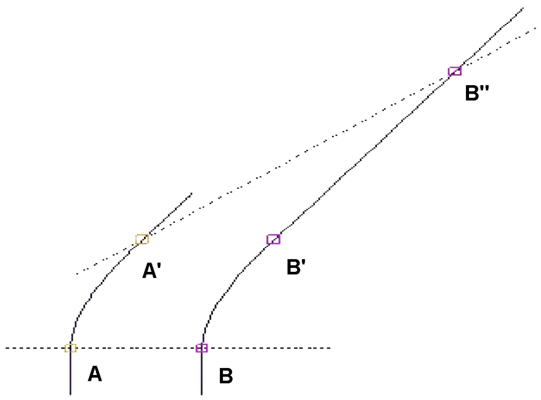

Fig. 1. Minkowski diagram for Bell's spaceship experiment.

Because the Lorentz interval is a geometric quantity that is independent of the choice of frame, we can compute its value in any frame which is computationally convenient, in this case frame S. Mathematically, in terms of the coordinates in frame S, we can represent the above statements by the following equations:

$$t_{B'} = t_{A'}\,, \tag{1}$$

$$x_{\mathrm{A}} - x_{\mathrm{B}} = x_{\mathrm{A}'} - x_{\mathrm{B}'} = L\,. \tag{2}$$

In frame S', $t'_{\mathrm{B}''} = t'_{\mathrm{A}'}$, so

$$t_{\mathrm{B}''} - \frac{v}{c^2} x_{\mathrm{B}''} = t_{A'} - \frac{v}{c^2} x_{\mathrm{A}'}\,. \tag{3}$$

Thus

$$t_{\mathrm{B}''} - t_{\mathrm{A}'} = \frac{v}{c^2}(x_{\mathrm{B}''} - x_{\mathrm{A}'})\,, \tag{4}$$

$$A'B'' = \sqrt{(x_{\mathrm{B}''} - x_{\mathrm{A}'})^2 - c^2((t_{\mathrm{B}''} - t_{\mathrm{A}'})} = (x_{\mathrm{B}''} - x_{\mathrm{A}'})\sqrt{1 - \frac{v^2}{c^2}}\,. \tag{5}$$

Past B' the rockets movement is uniform,

$$x_{\mathrm{B}''} - x_{\mathrm{B}'} = v(t_{\mathrm{B}''} - t_{\mathrm{B}'}) = v(t_{\mathrm{B}''} - t_{\mathrm{A}'}) = \frac{v^2}{c^2}(x_{\mathrm{B}''} - x_{\mathrm{A}'})\,, \tag{6}$$

$$x_{\mathrm{B}''} - x_{\mathrm{A}'} = (x_{\mathrm{B}''} - x_{\mathrm{B}'}) + (x_{\mathrm{B}'} - x_{\mathrm{A}'}) = \frac{v^2}{c^2}(x_{\mathrm{B}''} - x_{\mathrm{A}'}) + L\,, \tag{7}$$

$$x_{\mathrm{B}''} - x_{\mathrm{A}'} = \frac{L}{1 - v^2/c^2}\,. \tag{8}$$

Inserting (8) into (5), we obtain

$$A'B'' = \frac{L}{\sqrt{1 - v^2/c^2}}\,. \tag{9}$$

Thus, the distance between the spaceships has increased by the relativistic factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. It is interesting to note that we could have arrived to the same result by using the equations of length contraction derived for accelerated frames from the excellent Nikolić paper [3].

Objections and counter-objections have been published to the above analysis. For example, Paul Nawrocki [4] suggests that the string should not break, while Dewan [5] defends his original analysis from these objections in his reply. Bell [1] reported that he encountered much skepticism from "a distinguished experimentalist" when he presented the paradox. To attempt to resolve the dispute, an informal and non-systematic canvas was made of the CERN theory division. According to Bell, a "clear consensus" of the CERN theory division arrived at the answer that the string would not break. Bell further adds "Of course, many people who get the wrong answer at first get the right answer on further reflection" [1]. Later, Matsuda and Kinoshita [6] reported receiving much criticism after publishing an article on their independently rediscovered version of the paradox in a Japanese journal.

2. *Hyperbolic motion treatment*

We will now consider a more complex variant of the problem: find the distance between the rockets and the strain force in the connecting string as a function of time for uniformly accelerated rockets. The first part of the problem is best solved by using the equations of hyperbolic motion [7]. The second part of the problem is a complex issue that relates to a covariant formulation of the Hooke law [8]. The hyperbolic motion equation for the leading rocket as a function of the proper time τ and the proper acceleration $\mathbf{a}$ is

$$x_{\text{leading}} = d \cosh \frac{a\tau}{c}\,,$$
$$t_{\text{leading}} = \frac{d}{c} \sinh \frac{a\tau}{c}\,, \tag{10}$$
$$d = \frac{c^2}{a}\,.$$

The leading end of the string describes the same trajectory as x_{leading}. The trailing end describes the trajectory:

$$x_{\text{string}} = (d-L) \cosh \frac{a\tau}{c}\,,$$
$$t_{\text{string}} = \frac{(d-L)}{c} \sinh \frac{a\tau}{c}\,, \tag{11}$$

where L is the length of the string and also the rocket separation in the launcher frame S. If we consider the event A (x_{trailing}, t_{trailing}) the location of the tip of the trailing rocket at coordinate time t_{trailing}, then we can write the hyperbolic motion equation

$$x^2_{\text{trailing}} - c^2 t^2_{\text{trailing}} = d^2\,. \tag{12}$$

Since the distance between rockets in frame S is L, the corresponding point on the trajectory of the leading rocket (and the leading end of the string) at the same coordinate time t_{trailing} is (see Fig. 2)

$$(x_{\text{trailing}} + L)^2 - c^2 t^2_{\text{trailing}} = d^2\,. \tag{13}$$

Since

$$ct_{\text{trailing}} = x_{\text{trailing}} \tanh \frac{a\tau}{c}\,, \tag{14}$$

it follows immediately that

$$(x_{\text{trailing}} + L)^2 - d^2 = x^2_{\text{trailing}} \tanh^2 \frac{a\tau}{c}\,. \tag{15}$$

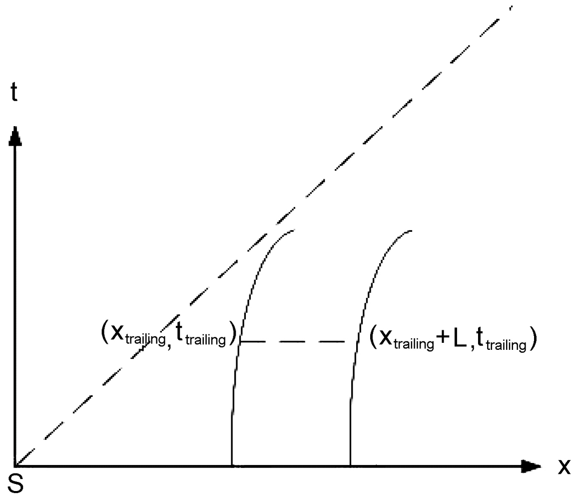

Fig. 2. Accelerated rocket's path.

The above equation gives the value for $x_{\rm trailing}$ as a function of the proper time τ

$$x_{\rm trailing} = \sqrt{d^2 + L^2 \sinh^2 \frac{a\tau}{c}} \; \cosh \frac{a\tau}{c} - L \cosh^2 \frac{a\tau}{c} \, . \tag{16}$$

The string stretch is expressed by the distance between the trailing end of the string and the tip of the trailing rocket (see Fig. 3),

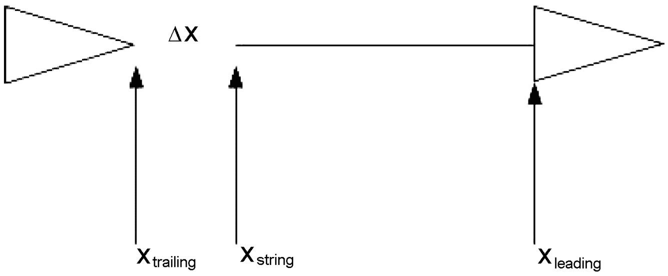

Fig. 3. The stretching of the string.

$$\Delta x = \sqrt{(x_{\rm string} - x_{\rm trailing})^2 - c^2 (t_{\rm string} - t_{\rm trailing})^2}$$

$$= \sqrt{(x_{\rm string} - x_{\rm trailing})^2 (1 - \tanh^2(a\tau/c))} = \frac{x_{\rm string} - x_{\rm trailing}}{\cosh(a\tau/c)} \, , \tag{17}$$

$$x_{\text{string}} - x_{\text{trailing}} = \left(d - L - \sqrt{d^2 + L^2 \sinh \frac{a\tau}{c}} \right) \cosh \frac{a\tau}{c} + L \cosh^2 \frac{a\tau}{c} \,. \tag{18}$$

So,

$$\Delta x = d - L - \sqrt{d^2 + L^2 \sinh^2 \frac{a\tau}{c}} + L \cosh \frac{a\tau}{c} \,. \tag{19}$$

Formula (19) gives the amount of string stretch as a function of proper time $\Delta x = \Delta x(\tau)$. Using it, we can determine the time elapsed after launch when the string will break. The string will break when

$$\Delta x = \frac{F_{\text{max}}}{k} \,, \tag{20}$$

where k is Young modulus. In the present paper, we have decided to use the Newtonian form of the Hooke law rather than resorting to the more modern covariant formulation [8]. There are two reasons for this: one, the present paper is a *didactical* paper, rather than a research in a new domain. The second, and more important reason, is that in the process of researching the subject we have discovered the Gron paper to be in error and we had to rework the covariant formulation of the Hooke law through a lengthy computation [9] that is beyond the scope of the current paper. Either way, at spaceship speeds significantly less than c, expression (20) holds with a very high precision.

$$\frac{F_{\text{max}}}{k} - d + L = -\sqrt{d^2 + L^2 \sinh \frac{a\tau}{c}} + L \cosh \frac{a\tau}{c} \,. \tag{21}$$

Using the notation

$$A = F_{\text{max}} - d + L \,, \tag{22}$$

we obtain immediately the proper time when the string breaks as

$$\tau = \frac{c}{a} \,\mathrm{Ar}\cosh \left(\frac{A^2 + L^2 - d^2}{2AL} \right) \,. \tag{23}$$

By using the equations of hyperbolic motion, we derived the variable distance between rockets as a function of proper time as well the time elapsed between rocket takeoff and string breaking.

3. Conclusions

We have produced a very simple and comprehensive solution for the Bell thought experiment. As we have demonstrated, using the equations of hyperbolic motion that we can easily derive the distance between rockets as a function of time as well the time elapsed between rocket takeoff and string breaking. The author expresses his gratitude for the valuable suggestions received from the two referees.

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ANALIZA BELLOVOG POKUSA S DVA BRODA HIPERBOLNIM GIBANJEM

U Bellovom zamišljenom "svemirskom" pokusu, dva su broda početno u mirovanju u zajedničkom inercijskom sustavu i povezani su napetim užetom. U nultom trenutku u tom sustavu, oba se broda počnu ubrzavati s jednakim stalnim vlastitim ubrzavanjem **a**, mjereno mjeračima ubrzanja na tim brodovima. Pita se: hoće li se uže prekinuti, tj., povećava li se razmak brodova? Predstavljamo dva pristupa, jedan s Minkowskijevim prostorno-vremenskim dijagramima, i drugi primjenom jednadžbi za ubrzano gibanje specijalne teorije relativnosti. Drugi pristup omogućuje računanje razmaka brodova kao i napregnutost užeta kao funkciju vlastitog vremena. Radi jednostavnosti, u ovom se razmatranju pretpostavlja da su brodovi i uže potpuno kruti, kako bi se izbjegli mali učinci na njihove duljine tijekom ubrzanja. Tema ovog Bellovog paradoksa često se nalazi u udžbenicima o teoriji relativnosti, ali potpuno i stvarno rješenje još nije objavljeno.