ISSN 1330–0016
CODEN FIZBE7

INHERENTLY RELATIVISTIC QUANTUM THEORY
PART III. QUANTIONIC ALGEBRA

EMILE GRGIN

Institute Ruđer Bošković, HR-10000 Zagreb, Croatia
E-mail address: eg137@ix.netcom.com

Dedicated to Professor Kseno Ilakovac on the occasion of his 70$^{\mathrm{th}}$ birthday

Received 7 January 2002; Accepted 18 February 2002
Online 6 April 2002

Quantum mechanics and relativity are not compatible at the structural level, and this makes it very difficult to unify them. The incompatibility might mean that a complete quantum theory unified with relativity exists, but is unknown, while standard quantum mechanics, as a special case, cannot be relativistic. If so, searching for generalizations is well justified, but the question is how. An old idea is to substitute a structurally richer algebra for the field of complex numbers, but such attempts have not brought the theory closer to relativity in the past. The present work is also based on this idea, but, unlike previous attempts, is not searching for new number systems among existing mathematical structures. From general considerations developed in the first two parts of this work, a new mathematical structure, referred to as *the quantionic algebra*, is derived as a theorem in the present paper. It is unique, manifestly relativistic, and generalizes the field of complex numbers in a manner consistent with quantum theory.

PACS numbers: 02.10.Jf, 03.65.-w UDC 530.145

Keywords: quaternion, octonion, division algebra, quantion, quantum, unification

1. *Introduction*

The research program undertaken in the four-part work, of which the present article is the third, is being essentially completed in the sequel with the formulation of a new mathematical structure, $\mathbb{D}$, that generalizes the field $\mathbb{C}$ of complex numbers.[1] We call it the *quantionic algebra*, and its elements *quantions*.[2] The general procedure consists in *extracting the quantionic algebra embedded in a given quantal algebra*. This statement is to be clarified and motivated before the technical work can begin in Section 2.

The quantal algebra $\{\mathcal{O}, \sigma, \alpha; e\}$, axiomatically defined in Ref. [2], is a two-product real algebraic structure satisfying the Jacobi, Leibnitz and association identities:

$$(f\alpha g)\,\alpha h + (g\alpha h)\,\alpha f + (h\alpha f)\,\alpha g \equiv 0, \tag{1}$$

$$f\alpha\,(g\sigma h) \equiv (f\alpha g)\,\sigma h + g\sigma\,(f\alpha h)\,, \tag{2}$$

$$[f, g, h] \equiv ag\alpha\,(h\alpha f)\,. \tag{3}$$

The element $e \in \mathcal{O}$ is the unit, $e\sigma f \equiv f$. This abstract structure contains three variants:

The case $a = 1$ corresponds to quantum mechanics. Additionally, in standard quantum mechanics the Lie substructure $\{\mathcal{O}, \alpha\}$ is semi-simple (i.e., the metric **is** non-singular) and central (meaning that there are no unnecessary constants, i.e., that $e\alpha f \equiv 0$ is the only such identity).

The case $a = 0$ corresponds to classical mechanics.

The case $a = -1$ corresponds to no physical theory.[3]

As there is no need in the sequel to refer to the general principles at the source of the identities (1), (2) and (3), we now take these identities, including the two restrictions stated for $a = 1$, to be the postulates of quantal algebra.

Intuitively, quantal algebra is the real algebra of quantum-mechanical observables stripped of their representation by Hermitian matrices. Its intended role is to provide a framework for seeking new versions of quantum mechanics that might be compatible with relativity. It was proved in Ref. [3] that if such versions exist, they can be found only among a few very specific candidates (the "non-unitary realizations" of quantal algebra). These possibilities are further pared down in this section to a single one (referred to as "the non-unitary quantal algebra"). This structure is built on the real underlying linear space $L\,(2, 4)$. Its two products, α and σ, represented by tensors of valence 3 in $L\,(2, 4)$, are derived in the next section.

[1] The last of these articles, to be published under the title "Inherently Relativistic Quantum Theory — Part IV, Quantionic Theorems", is a compilation of theorems that will be needed in physical applications of quantionic mathematics. The conceptual construction of quantions is completed in the present article.

[2] In the first paper on the subject, [1], this algebra was called "quantal ring", and its elements "quantals". By renaming it to "quantionic algebra" (on the model of "quaternionic algebra"), one avoids the risk of confusion with the abstract meaning of "quantal", as in "quantal algebra". Moreover, the term "quantion" (a noun, like "quaternion") is more appropriate a name for the elements of the algebra than "quantal" (an adjective, as in "real" number). As for "algebra" instead of "ring", both terms are acceptable, but the former is preferable for being extensively used in physics.

[3] But we remain open-minded to the possibility that the cases of $a = 0$ and $a = -1$ might have other interpretations in the generalization of quantum mechanics we are developing.

But there is more to quantum mechanics than its real algebra of observables. What distinguishes it fundamentally from classical physics is that it describes the world in terms of complex "amplitudes", rather than directly in terms of mathematical objects made of real variables with immediate intuitive and observational meaning. Two different views of the amplitudes suggest different approaches to the search for a generalization of quantum mechanics.[4] One attitude considers as essential *what the amplitudes structurally are* (they are complex numbers), the other *what the amplitudes functionally do* (they factorize the states).

Taking the field $\mathbb{C}$ of complex numbers as axiomatic, the first attitude leads to the search for generalizations of quantum mechanics in modifications of Hilbert space. It appears that only one such mathematically structural and physically meaningful modification is possible. It is based on Penrose's theory of twistors [4, 5]. Its limitations in physical applications stem from the indefiniteness of the norm. On the other hand, if one requires that the norm be positive-definite, Hilbert space seems to allow no modifications whatsoever. Whence the general conclusion that quantum mechanics and relativity are structurally incompatible.

The second attitude allow us to look for generalizations of quantum mechanics based on generalizations of its underlying complex number system. This idea is very old and suggests itself rather naturally, but, as an *ad hoc* idea, it comes with no guidelines and no "no-go theorems". Hence, all known mathematical structures that in some respect generalize the complex numbers (specifically, the other division algebras, Clifford algebras, and Grassmann algebras) are candidates *a priori* if they can be put to work. They have all been explored (the quaternions extensively), but did not yield any new interesting generalization of quantum theory.

In the present work we revisit the second attitude, but not by the trial and error approach. Our approach is based on the following requirements or observations:

(1) Quantal algebra is to be the abstract structure of any generalization of quantum mechanics. The reasons are given in Ref. [2].

(2) The generalization of the complex numbers we are seeking should preserve the Hilbert space structure. Thus, formal Hermiticity of matrices, $H^\dagger = H$, should also be meaningful in the generalization of the complex numbers.

(3) Since commutators and anticommutators of ordinary Hermitian matrices, interpreted as the products α and σ, respectively, satisfy the identities (2) to (3), so should the generalized Hermitian matrices.

(4) Taking this requirement to the lowest dimension, $n = 1$, it follows that a generalization of the complex numbers compatible with quantum mechanics must be a quantal algebra if expressed in terms of "real" and "imaginary" parts.

This last conclusion is both a "no-go" theorem[5] and a map to the discovery of the number system that works, if one exists — regardless of whether or not this

[4]By "generalization" we mean "new concrete version, structurally richer than the original". It is in this sense that we are seeking generalizations of the field of complex numbers and of quantum mechanics. Quantal algebra, on the other hand, is an "abstract generalization" of quantum mechanics.

[5]It immediately eliminates quaternions, octonions and Clifford algebras as candidate number systems for a new version of quantum mechanics.

system happens to be a known mathematical structure. Candidates are associative algebras that can be extracted out of concrete quantal algebras. We shall see that exactly one non-trivial solution exists.

To derive it, we exploit the idea of internal complexification introduced in Ref. [2]. We briefly review it as a sequence of conceptual steps.

(1) Given a concrete quantal algebra, identify in it, if any exists, all subalgebras isomorphic with the field $\mathbb{C}$ of complex numbers. This means finding all solutions $J \in \mathcal{O}$ of the equation $J\sigma J + e = 0$ (note: $J = \mathrm{i}e$ is not a solution, since we want J to be an observable, and observables are real). Clearly, if J is a solution, so is $J^* \overset{\mathrm{def}}{=} -J$. For such a J, the subalgebra isomorphic to $\mathbb{C}$ is $e\mathbb{R} \oplus J\mathbb{R}$, provided J and J^* are inequivalent, i.e., not related by a continuous transformation.

(2) A generalization of $\mathbb{C}$ now suggests itself as the centralizer of $e\mathbb{R} \oplus J\mathbb{R}$. The centralizer is the set of "algebraic constants", i.e., of all elements fixed under the group of automorphisms. In the quantal algebra $\mathcal{O}$, the centralizer is isomorphic to the field $\mathbb{R}$ of real numbers (the linear subspace $e\mathbb{R}$ spanned by the unit e). To expand it to $e\mathbb{R} \oplus J\mathbb{R}$, we are to find the centralizer of J, i.e., the linear subspace $\mathcal{O}_J \in \mathcal{O}$ of all observables f such that $J\alpha f = 0$. As it happens, $\mathcal{O}_J$ is also a quantal algebra, i.e., $\{\mathcal{O}_J, \sigma, \alpha; e, J\}$.[6] Hence, no further reduction of the set of observables is required.

(3) The quantal algebra $\{\mathcal{O}_J, \sigma, \alpha; e, J\}$ generalizes the field $\mathbb{C}$, but with a two-product structure, while $\mathbb{C}$ has a single associative product. To eliminate this objection, we define in $\mathcal{O}_J$ a new product,

$$f\beta g \overset{\mathrm{def}}{=} f\sigma g + J\sigma\left(f\alpha g\right),$$

for which one directly verifies, by the identities (2) to (3), that it is associative. Hence, the quantal generalization of the field $\mathbb{C}$ is the associative algebra $\{\mathcal{O}_J, \beta\}$.

(4) Since J commutes with all elements of $\mathcal{O}_J$, every element $f \in \mathcal{O}_J$ is of the form

$$f = f_r + J\beta f_i,$$

where the uniquely defined observables f_r, f_i are J-real, i.e., fixed under the operator $\mathcal{C}$. The eigenspaces of $\mathcal{C}$ are the linear subspaces $\mathcal{R}$ and $J\beta\mathcal{R}$ defined by the conditions

$$\begin{aligned} \mathcal{C} &: \mathcal{R} \to \mathcal{R} \\ \mathcal{C} &: J\beta\mathcal{R} \to -J\beta\mathcal{R} \end{aligned}$$

Clearly, $\dim\left(\mathcal{R}\right) = \dim\left(J\beta\mathcal{R}\right)$.

[6] Indeed, if $f, g, h \in \mathcal{O}_J$, then, by (1), $(f\alpha g) \in \mathcal{O}_J$, and, by (2), $(f\sigma g) \in \mathcal{O}_J$, so that $\mathcal{O}_J$ is stable under the quantal products σ, α.

(5) Generalize complex conjugation to the conjugation operator $\mathcal{C}$ defined by the mappings

$$\begin{aligned}\mathcal{C} &: e \longmapsto e, \\ \mathcal{C} &: J \longmapsto -J.\end{aligned}$$

It remains to be verified explicitly that $\mathcal{C}$ is a "true" involution in $\mathcal{O}_J$. The reason is that the algebra of observable, being much richer than $\mathbb{C}$, might contain a continuous group of transformations that could undo the action of $\mathcal{C}$, i.e., bring $-J$ back to J — in which case there would be no stable separation of objects into "real" and "imaginary".

We see from its construction that the algebra $\{\mathcal{O}_J, \beta\}$ is a generalization of the field of complex numbers with roots in quantum mechanics. We introduce the following definition:

Definition 1 *If, for a given quantal algebra $\{\mathcal{O}, \sigma, \alpha; e\}$, an associative algebra $\{\mathcal{O}_J, \beta; e, J\}$ exists, we refer to is as the **quantionic algebra** embedded in the given quantal algebra, and to its elements as **quantions**.*

The formal problem of generalizing quantum mechanics is now solved in principle. Obviously, to generalize the field $\mathbb{C}$ of complex numbers, a quantionic algebra must be richer than $\mathbb{C}$, i.e., not isomorphic with it. We call it "trivial" if it is. Thus, we will be searching for *non-trivial quantionic algebras*.

According to the classification of realizations, Ref. [3], all semi-simple quantal algebras fall into two classes: (1) The infinite family based on unitary Lie algebras, where $\{\mathcal{O}, \alpha\} = su\,(n)$ (referred to as *the unitary quantal algebras*), and, (2) the two quantal algebras for which $\{\mathcal{O}, \alpha\} = so\,(p, q)$, with $p + q = 3$ or $p + q = 6$ (referred to as *the non-unitary quantal algebras*). As standard quantum mechanics belongs to the unitary class and does not contain a generalization of $\mathbb{C}$,[7] generalizations of the theory are to be sought only among the non-unitary realizations.

Of the two candidates, the three-dimensional solution is eliminated by the following result:

Lemma 2 *The quantal algebra built over the Lie algebra $so\,(p, q)$, where $p+q = 3$, contains no non-trivial quantionic algebra.*

Proof. Let $\eta_{AB} = \mathrm{diag}\,\{1, -1, -1\}$ be the metric tensor in the representation space (reversing the signs has no structural effect), and let E_1, E_2, E_3 form an orthonormal basis. Then, $(E_1, E_1) = 1$, $(E_2, E_2) = (E_3, E_3) = -1$. The underlying linear space of the quantal algebra is $\mathcal{O} = \{e, e_1, e_2, e_3\}$, where $e_1 = E_2 \wedge E_3$, $e_2 = E_3 \wedge E_1$, $e_3 = E_1 \wedge E_2$, are the generators of the orthogonal group $SO\,(1, 2)$. The

[7] No square root of $-I$ can be a Hermitian matrix, since, for H Hermitian, $\mathrm{Tr}(H^2) > 0$, while $\mathrm{Tr}(-I) < 0$).

Lie algebra $\{\mathcal{O}, \alpha\}$ and the Jordan algebra $\{\mathcal{O}, \sigma\}$ are defined by the multiplication tables:

α	e_1	e_2	e_3
e_1	0	$-e_3$	e_2
e_2	e_3	0	e_1
e_3	$-e_2$	$-e_1$	0

,

σ	e_1	e_2	e_3
e_1	e	0	0
e_2	0	$-e$	0
e_3	0	0	$-e$

.

Thus, a square root of $-e$ exists, for example, $J = e_3$. But since only e and J commute with J, the quantionic algebra is $\mathcal{O}_J = \{e, J\}$. It is isomorphic to $\mathbb{C}$ — hence trivial. ■

This leaves only the six-dimensional case as a candidate. We analyze it in the next section.

2. *The quantal algebra* $\mathcal{O}\,(2,4)$

Let η_{AB} denote the metric tensor in a 6-dimensional linear space V^6. Working in an orthonormal basis, we shall also use the notation $\eta_A \overset{\text{def}}{=} \eta_{AA}$. In terms of an orthonormal hexad $\{E_A\} = \{E_0, E_1, ..., E_5\}$ in V^6, the generators of the 15-dimensional orthogonal group are the exterior products of basis vectors, i.e., they are the simple bi-vectors,

$$e_{AB} \overset{\text{def}}{=} E_A \wedge E_B. \tag{4}$$

As proved in Ref. [3], they form, together with the unit e, the real 16-dimensional space, $\mathcal{O}$, of a quantal algebra $\{\mathcal{O}, \sigma, \alpha\}$. But the proof in question did not generate the algorithm for the Jordan product σ (the Lie product α is known). It only guaranteed its existence. We derive it in this section.

The Lie product in the subalgebra $\{\mathcal{O}, \alpha\}$ is given by the universal expression

$$e_{PQ}\ \alpha\ e_{RS} = \eta_{PR}\ e_{QS} - \eta_{QR}\ e_{PS} - \eta_{PS}\ e_{QR} + \eta_{QS}\ e_{PR}. \tag{5}$$

We are now to derive the expression for the Jordan product σ. In the next three lemmata, the six labels $A, B, \cdots, F$ from the beginning of the alphabet are assumed to be different, the lexicographic order is defined as positive, and summation over repeated labels is not used.

Lemma 3 *If the Lie product of two generators is different from zero, their Jordan product vanishes, i.e.,*

$$e_{XA}\ \sigma\ e_{XB} = 0. \tag{6}$$

Proof. According to the expression (5), only the pairs of generators that have one label in common have a non-vanishing Lie product. Specifically,

$$e_{XA}\ \alpha\ e_{XB} = \eta_X\ e_{AB}.$$

The Jacobi product $e_{XA}\sigma e_{XB}$, being symmetric in the free labels AB, can only be of the form

$$e_{XA}\ \sigma\ e_{XB} = f_{XAB}\ e,$$

for some real coefficients $f_{XAB} = f_{XBA}$. Computing the associator

$$[e_{XA}, e_{XB}, e_{XC}] = a\ e_{XB}\ \alpha\,(e_{XC}\ \alpha\ e_{XA})$$

for A, B, C, X all different yields

$$f_{XAB}\ e_{XC} - f_{XBC}\ e_{XA} = a\ \eta_X\ e_{XB}\ \alpha\ e_{CA} = 0.$$

Since e_{XC} and e_{XA} are basis elements in $\mathcal{O}$, the only solution is, $f_{XAB} = 0$. Hence, $e_{XA}\ \alpha\ e_{XB} = 0$, proving the statement. ■

Lemma 4 *The square of a generator is proportional to the unit,*

$$e_{AB}\ \sigma\ e_{AB} = a\ \eta_A\ \eta_B\ \ e. \tag{7}$$

Proof. The expression $e_{XY}\ \alpha\ \ (e_{AB}\ \sigma\ e_{AB})$ vanishes unless the pairs XY and AB have exactly one label in common, in which case it vanishes by the identity (2):

$$e_{AC}\ \alpha\ \ (e_{AB}\ \sigma\ e_{AB}) = 2\ \eta_A\ e_{CB}\ \sigma\ e_{AB} = 0.$$

Hence, for every $f \in \mathcal{O}$, $f\alpha\,(e_{AB}\ \sigma\ e_{AB}) = 0$, implying $e_{AB}\ \sigma\ e_{AB} = h_{AB}\ e$ for some real constant h_{AB}. Computing the associator $[e_{AB}, e_{AB}, e_{AC}]$ in both ways, by its definition and by relation (3), one obtains

$$h_{AB}\ e_{AC} = a\,\eta_A\ e_{AB}\ \alpha\ e_{CB} = a\,\eta_A\ \eta_B\ e_{AC}.$$

This implies $h_{AB} = a\,\eta_A\ \eta_B$, proving the statement. ■

Lemma 5 *The Jordan product of generators without common label is proportional to the unique generator which has no common label with either factor. Specifically,*

$$e_{AB}\ \sigma\ e_{CD} = \Theta\,\eta_E\,\eta_F\,e_{EF}, \tag{8}$$

where $\Theta = \pm 1$.

Proof. Relation (3) implies $(e_{AB}\ \sigma\ e_{CD})\,\sigma\ e_{EF} = e_{AB}\,(\sigma\ e_{CD}\ \sigma\ e_{EF})$, which can be satisfied only if $e_{AB}\ \sigma\ e_{CD} = k_{EF}\ e_{EF}$, where k_{EF} is an unknown real coefficient. The same relation also implies $(e_{AB}\ \sigma\ e_{AB})\,\sigma\ e_{CD} = e_{AB}\ \sigma\ \ (e_{AB}\ \sigma\ e_{CD})$, which expands to

$$a\ \eta_A\ \eta_B\ e_{CD} = k_{EF}\ e_{AB}\ \sigma\ e_{EF} = k_{EF}\ \ \Theta\eta_C\ \eta_D\ e_{CD},$$

implying $k_{EF}\, k_{CD}\, \eta_A\, \eta_B = a$. Symmetry under all permutations of the labels yields $k_{AB} = \Theta\, \eta_A\, \eta_B$ and $\eta_A\, \eta_B\, \eta_C\, \eta_D\, \eta_E\, \eta_F = a$. The first of these two equations proves relation (8). The second, obtained in passing, is important enough to be stated separately as theorem Theorem 7. ■

Collecting these results, on obtains the following general expression for the Jordan product (we now use the summation convention):

Theorem 6 *Without restrictions on the labels, the Jordan product of two generators is*

$$e_{PQ}\ \sigma\ e_{RS} = \frac{1}{2}\, \Theta \varepsilon_{PQRSTU}\, e^{TU} + a\ (\eta_{PR}\ \eta_{QS} - \eta_{PS}\ \eta_{QR})\, e. \tag{9}$$

Proof. We have $\eta_E\, \eta_F\, e_{EF} = e^{EF}$, while the Levi-Civitta symbol collects the results (6), (7) and (8). The factor 1/2 compensates for the summation of two equal terms. ■

Theorem 7 *The determinant of the metric tensor is equal to the association parameter a,*

$$\det(\eta_{AB}) = a. \tag{10}$$

Proof. Clearly, $\det(\eta_{AB}) = \eta_A\, \eta_B\, \eta_C\, \eta_D\, \eta_E\, \eta_F = a$, obtained in proving Lemma 5. ■

This last result implies the following *fundamental theorem of quantionic algebra*:

Theorem 8 *There exists exactly one physical non-trivial non-unitary quantal algebra. It is* $\mathcal{O}(2,4)$.

Proof. Referring to the introduction, there is only one candidate. It is based on the orthogonal Lie algebras in six dimensions. Taking each assertion of the theorem in turn, the *existence* of quantal algebras $\mathcal{O}(p,q)$, with $p+q=6$, has been proved by exhibiting the expressions for the Lie and Jacobi products, relations (5), (9). Of these, two are physical, i.e., characterized by $a=1$. By Theorem 7, they are $\mathcal{O}(6,0)$ and $\mathcal{O}(2,4)$. The other two possibilities, with p and q exchanged, are structurally indistinguishable. Of these solutions, only $\mathcal{O}(2,4)$ is *non-unitary*, the reason being that $SO(6)$ is isomorphic to $SU(4)$. To prove that $\mathcal{O}(2,4)$ is *non-trivial*, we need to exhibit an observable J, such that $J\sigma J = -e$, whose centralizer is not isomorphic with $\mathbb{C}$, i.e., contains at least one element linearly independent of e and J. Clearly, there are many solutions. For example, taking $J = e_{AB}$ for A and B such that $\eta_A \eta_B = -1$, we see that e_{CD} is at least one element of the centralizer linearly independent of e and J. Hence, $\mathcal{O}(2,4)$ is non-trivial. ■

Having established that the pseudo-orthogonal group $SO(2,4)$ is subjacent to the unique physical non-trivial quantal algebra $\mathcal{O}(2,4)$, we note that this group already has a well-known physical meaning. It is the invariance group of the conformal compactification of Minkowski space, $\mathcal{M}^4$, discovered and investigated by

Roger Penrose [4]. It is remarkable that a structure, meaning $SO(2,4)$, with historical origin in strictly geometric considerations rooted in relativity theory should also make its appearance as the unique solution to strictly algebraic considerations rooted in quantum theory. This bolsters our expectation that the quantal approach ought to lead to a structural unification of quantum theory and relativity.

3. *The quantionic algebra*

In the next subsections we shall extract all quantionic algebras from the quantal algebra $\mathcal{O}(2,4)$ by following the strategy outlined in the Introduction.

3.1. Complex conjugation

As the ordinary conjugation of complex numbers is not directly transferrable to real linear spaces, we are to find an equivalent defining property that can be applied to the space $L(2,4)$.

The operator $\mathcal{C}$ in the algebra $\mathcal{O}(2,4)$ is meant to be the analogue of ordinary complex conjugation, but since the algebra $\mathcal{O}(2,4)$ is real, we will need an interpretation of conjugation in terms of real linear concepts. To recognize it in the field of complex numbers, consider an algebra $\mathcal{O}(2)$ spanned by the unit matrix and the antisymmetric matrix (the single generator of rotations in a real 2-dimensional linear space, L^2):

$$\mathcal{O}(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{R} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbb{R}.$$

Clearly, $\mathcal{O}(2)$ is isomorphic to the field $\mathbb{C}$ of complex numbers. The operator $\mathcal{C}$ is represented by matrix transposition. Since the unit antisymmetric matrix in two dimensions can be written as a bivector, say $\vec{m} \wedge \vec{n}$, the vectors $\vec{m}$ and $\vec{n}$ being orthonormal in L^2, one can also write

$$\mathcal{O}(2) = e\mathbb{R} \oplus (\vec{m} \wedge \vec{n})\,\mathbb{R}.$$

In this formalism, $\mathcal{C}$ is represented by a *mirror reflection* in the plane L^2, for example

$$\begin{aligned} \mathcal{C} \; &: \; \vec{m} \longmapsto -\vec{m}, \\ \mathcal{C} \; &: \; \vec{n} \longmapsto \vec{n}. \end{aligned}$$

The vectors $\vec{m}$ and $\vec{n}$ are uniquely defined as the eigenvectors of the operator $\mathcal{C}$, but there is an infinity of such operators mutually equivalent under rotations in L^2. By selecting the vectors $\vec{m}$ and $\vec{n}$ first, one uniquely specifies the conjugation.

This is the interpretation of complex conjugation we shall use in the algebra $\mathcal{O}(2,4)$. Since $\mathcal{O}(2,4)$ and $L^{(2,4)}$ are, respectively, the generalizations of $\mathcal{O}(2)$

and L^2, the conjugation operator $\mathcal{C}$ in $\mathcal{O}(2,4)$ will be defined in terms of mirror reflections in $L^{(2,4)}$. Our next task is to identify the operator $\mathcal{C}$ within the group of discrete transformations in $L^{(2,4)}$.

3.2. The conditions defining $J \in \mathcal{O}(2,4)$

To extract a quantionic algebra $\mathcal{O}_J(2,4)$ from the non-unitary quantal algebra $\mathcal{O}(2,4)$, one needs an element $J \in \mathcal{O}(2,4)$ which, as reviewed in the Introduction, is characterized by the following properties: It is a real square-root of minus unity,

$$J\sigma J = -e, \tag{11}$$

and behaves like an imaginary unit with respect to a true conjugation operator $\mathcal{C}$, i.e.,

$$\mathcal{C} \quad : \quad e \longmapsto e^* = e, \tag{12}$$

$$\mathcal{C} \quad : \quad J \longmapsto J^* = -J. \tag{13}$$

By “true”, we mean that the action of the operator $\mathcal{C}$ cannot be undone by a continuous transformation from the group $SO(2,4)$.

In addition to these conditions on J, an additional condition is given by the following lemma:

Lemma 9 *The quantionic imaginary unit J is a simple bivector,*

$$J = V \wedge W\,, \tag{14}$$

where $V, W \in L^{(2,4)}$ are mutually orthogonal vectors — one of positive unit norm, the other of negative unit norm.

Proof. Reminder: A simple bivector (or special bivector) is the exterior product of two vectors. Thus, the generators e_{AB}, defined by expression (4), are simple bivectors. A general bivector is a sum of simple bivectors. In six dimensions, referring to the linear space $L^{(2,4)}$, a bivector is either simple, or a sum of at most three simple bivectors. We shall prove that J is a simple bivector.

By definition, J is an observable, hence, a linear combination of the 16 basis observables e, e_{RS}, i.e., $J = xe + \sum_{R,S} u_{RS}\, e_{RS}$, where x and u_{RS} are real coefficients. One first observes that J cannot contain e, as conditions (12) and (13) don't mix. Hence, $J = \sum_{R,S} u_{RS}\, e_{RS}$ is the most general expression. Computing the square of this expression by substitution into the definition (9) of σ, one gets (using the summation convention),

$$\begin{aligned} J\sigma J &= u^{AB}u^{CD}e_{AB}\;\sigma\; e_{CD} \\ &= \frac{1}{2}\,u^{AB}u^{CD}\varepsilon_{ABCDTU}\,e^{TU} + u^{AB}u_{AB}\;e. \end{aligned}$$

Condition (11) implies $u^{AB}u_{AB} = -1$ and $u^{AB}u^{CD}\varepsilon_{ABCDTU} = 0$. This last relation further implies that all e_{AB} in the expansion of J have one label in common, say A. Thus,

$$J = E_A \wedge \sum_R u^{AR} E_R,$$

without summation over A. Since the sum is a vector, J is a simple bivector. Hence, if J exists, there exist in $L^{(2,4)}$ two vectors, say V and W (which can be taken to be orthogonal, $(V, W) = 0$), so that J is of the form (14), i.e., it is a simple bivector. The first relation, $u^{AB}u_{AB} = -1$, implies that one vector is of positive, the other of negative norm, e.g., $(V, V) = -1$, $(W, W) = 1$. ■

In the next section we analyze the group of discrete transformations in $L^{(2,4)}$, as we need it to derive the solutions for J.

3.3. The $\mathcal{CPT}$ group

The orthogonal transformations in $L^{(2,4)}$ consist of the group $SO\,(2, 4)$ of continuous transformations ("rotations" for short) and of a finite group of discrete transformations, i.e., *transformations that cannot be undone by rotations.* Two permanencies are at play in separating these two groups: The sign of the determinant, and the sign of the norm. Hence, there are two independent reflections in $L^{(2,4)}$: The reversal of a vector, $P \in L^{(2,4)}$ of positive norm, $(P, P) = 1$, and the reversal of a vector, $N \in L^{(2,4)}$ of negative norm, $(N, N) = -1$. Independence is expressed as orthogonality, $(P, N) = 0$. Anticipating the interpretation of these reflections (established at the end of this section), we denote them by $\mathcal{T}$ and $\mathcal{P}$, respectively:

$$\mathcal{T} \quad : \quad P \longmapsto -P, \tag{15}$$

$$\mathcal{P} \quad : \quad N \longmapsto -N. \tag{16}$$

They are involutions, $\mathcal{T}^2 = \mathcal{P}^2 = I$, and their product $\mathcal{TP} = \mathcal{PT}$, is the only additional metric-related discrete transformation.[8] We denote it by $\mathcal{C}$, i.e., $\mathcal{C} \overset{\text{def}}{=} \mathcal{TP}$. Clearly,

$$\mathcal{CPT} = I.$$

The discrete transformations form the dihedral group, $\{I, \mathcal{C}, \mathcal{P}, \mathcal{T}\}$.

We shall now distinguish the two orthonormal vectors P and N, i.e., consider them fixed in $L^{(2,4)}$. This splits the space $L^{(2,4)}$ into two parts: a 2-plane $L^{(1,1)}$, spanned by P and N, and its orthogonal complement $L^{(1,3)}$ — which is a linear

[8] In addition to these metric-related discrete transformations, there is an additional one which changes the orientation of the space, i.e., the definition of the measure. Let's denote it by $\mathcal{E}$. Linear objects (scalars, vectors and tensors) that change sign under the action of $\mathcal{E}$ are referred to as pseudo-objects. We now disregard this transformation, as it plays no role in the derivation of J.

Minkowski space. We refer to this splitting as *the Lorentz splitting*, for it extracts the Lorentz group from the orthogonal group $O(2,4)$.

In addition, we shall need a second splitting of the space $L^{(2,4)}$ into two subspaces of definite metric, L^2 and L^4. We call it the *structural splitting.* To define L^2, we use the already distinguished vector P. The second defining vector may be selected arbitrarily on the two-sheeted hyperboloid in the Minkowski space $L^{(1,3)}$. We denote it by Ω, and refer to it as *the structure vector*. Thus, $(\Omega,\Omega) = 1$ and $(\Omega, P) = 0$. The hyperplane in $L^{(1,3)}$ orthogonal to Ω is a three-dimensional Euclidean space. We denote it by Π, and refer to it as *the structure space.* Consequently, the subspace L^4 is spanned by N and Π. While the structural splitting is far from unique, it is completely determined once the structure vector Ω has been selected.

These two independent splittings are illustrated together in the following diagram:

	$L^{(1,3)}$	$L^{(1,1)}$
L^2	Ω	P
L^4	Π	N

Returning to the reflections, we note that $\mathcal{T}$, defined by (15), is equivalently defined as

$$\mathcal{T} : \Omega \longmapsto -\Omega. \tag{17}$$

The reason is that P and Ω are related by a continuous transformation from $SO(2,4)$, specifically, a rotation in L^2. Similarly, by a rotation in L^4, $\mathcal{P}$ can be equivalently defined as a reversal of any vector in Π. But since L^4 is even-dimensional, all vectors in Π can be reversed simultaneously — which is more elegant, as it does not require selecting for reversal any particular vector in Π. Hence, $\mathcal{P}$, defined by (16), is equivalently defined as

$$\mathcal{P} : \Pi \longmapsto -\Pi. \tag{18}$$

With the redefinitions (17) and (18), the discrete group $\{I, \mathcal{C}, \mathcal{P}, \mathcal{T}\}$ has been transferred to the linear Minkowski space $L^{(1,3)}$ — making it co-resident with the Lorentz group $SO(1,3)$. The transfer required the introduction of the structure vector Ω. Lets discuss the reason.

Time reversal is intuitively defined as the reversal of the flow of time. This is well-defined in non-relativistic physics, where time is absolute, and is also meaningful in the affine Minkowski space, where the time direction is global. But the Minkowski space $L^{(1,3)}$ we have is linear, not affine (i.e., it is not a Riemannian manifold that happens to be flat). Thus, there is no externally defined direction of time that $L^{(1,3)}$ could inherit. By selecting the structure vector Ω, we introduce a time direction.

Given Ω, an $\{\Omega, \Pi\}$-frame is automatically defined (with whatever basis in Π), and quantionic relations are often conveniently written in this frame. But Ω and

Π are themselves defined in the arbitrary basis tetrad $\{E_\alpha\}$, so that covariance is not lost. Since the frames $\{E_\alpha\}$ and $\{\Omega, \Pi\}$ will be used interchangeably, we shall need an appropriate terminology to refer to both. A frame $\{E_\alpha\}$ will be called a *Lorentz frame*, and, as usual, properly formed tensorial expressions in it will be said to be *covariant*. The name *structural frame* will be used for an $\{\Omega, \Pi\}$-frame, and properly formed quantionic expressions in it will be said to be *structural*.

3.4. The covariant formalism

The splitting of the linear space $L^{(2,4)}$ into the pair of vectors N, P and the linear Minkowski space $L^{(1,3)}$ induce a corresponding splitting of the quantal tensor algebra. With Greek indices running from 0 to 3, we rename the basis of observables e_{AB} as follows:

$$\left.\begin{aligned} e_{\alpha\beta} &\stackrel{\text{def}}{=} E_\alpha \wedge E_\beta\,, \\ n_\alpha &\stackrel{\text{def}}{=} N \wedge E_\alpha\,, \\ p_\alpha &\stackrel{\text{def}}{=} P \wedge E_\alpha\,, \\ j &\stackrel{\text{def}}{=} N \wedge P\,. \end{aligned}\right\} \tag{19}$$

The transcription of the expression (5) for the Lie product into this covariant formalism leads to the following expressions (listing only the non-vanishing products):

$$\left.\begin{aligned} e_{\alpha\beta}\ \alpha\ e_{\gamma\delta} &= \eta_{\alpha\gamma}\ e_{\beta\delta} - \eta_{\alpha\delta}\ e_{\beta\gamma} - \eta_{\beta\gamma}\ e_{\alpha\delta} + \eta_{\beta\delta}\ e_{\alpha\gamma}\,, \\ e_{\alpha\beta}\ \alpha\ n_\gamma &= \eta_{\alpha\gamma}\ n_\beta - \eta_{\beta\gamma}\ n_\alpha\,, \\ e_{\alpha\beta}\ \alpha\ p_\gamma &= \eta_{\alpha\gamma}\ p_\beta - \eta_{\beta\gamma}\ p_\alpha\,, \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} n_\alpha\ \alpha\ n_\beta &= -e_{\alpha\beta}\,, \\ p_\alpha\ \alpha\ p_\beta &= e_{\alpha\beta}\,, \\ n_\alpha\ \alpha\ p_\beta &= \eta_{\alpha\beta}\ j\,, \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} n_\alpha\ \alpha\ j &= p_\alpha\,, \\ p_\alpha\ \alpha\ j &= n_\alpha\,. \end{aligned}\right\} \tag{22}$$

To transcribe the Jordan product (9), we specify as $\{E_0, E_1, E_2, E_3, N, P\}$ the positive orientation of the basis vectors in $L^{(2,4)}$, thus identifying N and P with the fourth and fifth basis vectors. The non-vanishing entries for this product are given by the following relations:

$$\left.\begin{aligned} e_{\alpha\beta}\ \sigma\ e_{\gamma\delta} &= \varepsilon_{\alpha\beta\gamma\delta}\ j + (\eta_{\alpha\gamma}\ \eta_{\beta\delta} - \eta_{\alpha\delta}\ \eta_{\beta\gamma})\, e\,, \\ e_{\alpha\beta}\ \sigma\ n_\gamma &= \varepsilon_{\alpha\beta\gamma\delta}\ p^\delta\,, \\ e_{\alpha\beta}\ \sigma\ p_\gamma &= \varepsilon_{\alpha\beta\gamma\delta}\ n^\delta\,, \\ e_{\alpha\beta}\ \sigma\ j &= -\tfrac{1}{2}\ \varepsilon_{\alpha\beta\gamma\delta}\ e^{\gamma\delta} = -\,{}^*e_{\alpha\beta}\,, \end{aligned}\right\} \tag{23}$$

$$\left.\begin{array}{c} n_\alpha \ \sigma \ n_\beta = -\eta_{\alpha\beta} \ e \,, \\ p_\alpha \ \sigma \ p_\beta = \eta_{\alpha\beta} \ e \,, \\ n_\alpha \ \sigma \ p_\beta = -\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} \ e^{\gamma\delta} = - {}^*e_{\alpha\beta} \,. \\ j \, \sigma \, j = -e \,. \end{array}\right\} \tag{24}$$

These two sets of multiplication rules now supersede the general algorithms (5) and (9).

This completes the preparations needed to identify all possible solutions for J, encapsulated in the following theorem:

Theorem 10 *The most general expression for J is*

$$J = \Omega^\rho n_\rho, \tag{25}$$

where the coefficients Ω^ρ are the components of the structure vector Ω,

$$\Omega = \Omega^\rho E_\rho. \tag{26}$$

Proof. Taking (17) and (18) as the definitions of $\mathcal{T}$ and $\mathcal{P}$, the conjugation $\mathcal{C}$ reverses Ω, and Π,

$$\mathcal{C} \ : \ \Omega \to -\Omega,$$
$$\mathcal{C} \ : \ \Pi \to -\Pi,$$

but does not affect P and N. Hence, only two expressions for J satisfy the requirements of Lemma 9 and of conjugation

$$\mathcal{C} : J \to -J. \tag{27}$$

They are

$$J = N \wedge \Omega \tag{28}$$

and

$$J = P \wedge \pi,$$

where π is an arbitrary unit space-like vector, $\pi \in \Pi$. This latter expression drops out, however, because it is not a true involution, i.e., J and $-J$ are equivalent, in the sense of being related by a continuous Lorentz transformation (because π and $-\pi$ are on the single sheeted unit hyperboloid, which is an orbit of the Lorentz group). By contrast, Ω is on the double sheeted unit hyperboloid, so that expression 28 is a true involution. It is the only solution for J.

Expressing Ω in the Lorentz frame, expression (12), one obtains

$$J = N \wedge \Omega^\rho E_\rho = \Omega^\rho n_\rho, \tag{29}$$

which proves the relation (11). ■

3.5. The centralizer of J

Once $J \in \mathcal{O}(2,4)$ has been specified by fixing the vector Ω, one derives the centralizer $\mathcal{O}_J(2,4)$ by expressing the general observable $f \in \mathcal{O}(2,4)$ as a linear combination of the basis observables introduced above, and then imposing the condition $J\alpha f = 0$. The linear space of coefficients which satisfy this condition identically is, by definition, the underlying linear space of the quantionic algebra. The products in $\mathcal{O}_J(2,4)$ follow from the expressions for the products in $\mathcal{O}(2,4)$.

With upper case letters denoting the coefficients (scalars, vectors, and tensor), the expansion for the most general observable f is

$$f = Ae + Bj + V^\mu n_\mu + W^\mu p_\mu + T^{\mu\nu} e_{\mu\nu}. \tag{30}$$

To impose the condition $(\Omega^\rho n_\rho)\,\alpha f = 0$, one considers each term separately. Clearly, A is arbitrary. We compute the other coefficients in turn using the relations (20) to (22).

Computing B : The condition

$$(\Omega^\rho n_\rho)\,\alpha\,(Bj) = B\Omega^\rho p_\rho = 0$$

implies $B = 0$.

Computing $V^\mu n_\mu$: The condition

$$(\Omega^\rho n_\rho)\,\alpha\,(V^\mu n_\mu) = \Omega^\rho V^\mu\,(n_\rho \alpha n_\mu) = -\Omega^\rho V^\mu e_{\rho\mu} = 0$$

implies that V^μ is proportional to Ω^μ. Hence,

$$V^\mu n_\mu = VJ\,,$$

where V is an arbitrary scalar.

Computing $W^\mu p_\mu$: The condition

$$(\Omega^\rho n_\rho)\,\alpha\,(W^\mu p_\mu) = \Omega^\rho W^\mu\,(n_\rho \alpha p_\mu) = (\Omega, W)\,j = 0$$

implies that W^μ is an arbitrary vector in Π.

Computing $T^{\mu\nu} e_{\mu\nu}$: The condition

$$(\Omega^\rho n_\rho)\,\alpha\,(T^{\mu\nu} e_{\mu\nu}) = \Omega^\rho T^{\mu\nu}\,(n_\rho \alpha e_{\mu\nu}) = 2\Omega^\rho T^{\mu\nu} \eta_{\rho\nu} n_\mu = 0$$

implies that the most general solution is

$$T^{\mu\nu} e_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\sigma\tau} \Omega_\sigma T_\tau e_{\mu\nu}\,,$$

where T^ν is an arbitrary vector in Π (the factor 1/2 is cosmetic).

Collecting these partial results, we see that the most general quantion $f \in \mathcal{O}_J(2,4)$ is of the form

$$f = Ae + V\Omega^\mu n_\mu + W^\mu p_\mu + \frac{1}{2}\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma T_\tau e_{\mu\nu}\,. \tag{31}$$

We now observe that the first and last terms are invariant under conjugation. The first by definition, since A is a real number. In the last term, the direction represented by each index is reversed (σ by $\mathcal{T}$, the other three by $\mathcal{P}$, as they belong to Π), so that the eigenvalue of $\mathcal{C}$ is $(-1)^4 = 1$. Hence, these two terms belong to the J-real part $\mathcal{R}$ of $\mathcal{O}(2,4)$:

$$Ae + \frac{1}{2}\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma T_\tau e_{\mu\nu} \in \mathcal{R}\,. \tag{32}$$

The second and third terms in expression (31) are reversed by conjugation (the second is reversed by $\mathcal{T}$ due to the presence of Ω, the third by $\mathcal{P}$ because $W^\mu \in \Pi$). Thus, these terms are J-imaginary. We still have to verify, as a matter of consistency, that they belong to the J-imaginary part $J\beta\mathcal{R}$ of $\mathcal{O}(2,4)$. For $V\Omega^\mu n_\mu$, which, by relation (25), is equal to VJ, this is evident. For the second term, the question is whether a vector X_τ exists such that

$$W^\mu p_\mu = \frac{1}{2}J\beta\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma X_\tau e_{\mu\nu}\,.$$

Let's expand $J\beta\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma e_{\mu\nu}$ using the second of relations (23):

$$\begin{aligned}\frac{1}{2}J\beta\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma e_{\mu\nu} &= \frac{1}{2}\Omega^\rho n_\rho\beta\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma e_{\mu\nu} = \frac{1}{2}\Omega^\rho\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma\left(n_\rho\beta e_{\mu\nu}\right)\\ &= \frac{1}{2}\Omega^\rho\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma\varepsilon_{\mu\nu\rho\delta}p^\delta = \left(\delta^\sigma_\delta\delta^\tau_\rho - \delta^\sigma_\rho\delta^\tau_\delta\right)\Omega^\rho\Omega_\sigma p^\delta\\ &= \left(\Omega^\tau\Omega_\delta - \delta^\tau_\delta\right)p^\delta.\end{aligned} \tag{33}$$

Hence

$$W_\delta = \left(\Omega^\tau\Omega_\delta - \delta^\tau_\delta\right)X_\tau\,, \tag{34}$$

but since $X_\tau \in \Pi$, the second term vanishes due to $\Omega^\tau X_\tau = 0$. Hence, $X_\tau = -W_\tau$. Consequently,

$$V\Omega^\mu n_\mu + W^\mu p_\mu \in J\beta\mathcal{R}\,.$$

This verifies that the quantionic algebra $\mathcal{O}_J(2,4)$ splits into a J-real and a J-imaginary part,

$$\mathcal{O}_J(2,4) = \mathcal{R} \oplus J\beta\mathcal{R}\,, \tag{35}$$

in analogy with the field of complex numbers.

We can now prove the key theorem of inherent unification:

Theorem 11 *The linear space of real coefficients spanning the linear space $\mathcal{R}$ of real quantions is isomorphic to the linear Minkowski space, M_0.*

Proof. By relations (32), the J-real quantions are of the form

$$u = Ue + \frac{1}{2}\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma u_\tau e_{\mu\nu}\,, \tag{36}$$

where $U \in \mathbb{R}$ and $u^\mu \in \Pi$, i.e., $\Omega_\mu u^\mu = 0$. But this last condition need not be imposed, as the Ω-component of u^μ is destroyed in the construction $\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma u_\tau$. Thus, by allowing u_τ to be an arbitrary 4-vector, and defining U as $U = \Omega_\mu u^\mu$, one obtains the following relation between $\mathcal{R}$ and M_0 :

$$u = \left(e\Omega^\tau + \frac{1}{2}\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma e_{\mu\nu}\right)u_\tau\,. \tag{37}$$

This proves the isomorphism $\mathcal{R}u \leftrightarrow u_\tau \in M_0$. ■

It will prove convenient to have a simple symbol, ω^τ, for the hybrid object (observable and 4-vector) that defines the mapping (37):

$$\omega^\tau \stackrel{\text{def}}{=} e\Omega^\tau + \frac{1}{2}\varepsilon^{\sigma\tau\mu\nu}\Omega_\sigma e_{\mu\nu}\,. \tag{38}$$

Then, $u = \omega^\tau u_\tau$.

By taking $u = e$ in relation (37) and solving for u_μ, one obtains $u_\mu = \Omega_\mu$. Hence, the unit $e \in \mathcal{R}$ is represented in M_0 by the vector Ω^μ. Let's emphasize this new insight:

The algebraic unit $e \in \mathcal{O}_J$ and the time direction $\Omega^\mu \in L^{(1,3)}$ are the same object. From the algebraic, or quantum mechanical, viewpoint, this object manifests itself as the unit. From the geometric, or relativistic, viewpoint it manifests itself as the direction of time.

It is this total merging in the vector Ω of two fundamentally different concepts that justifies calling Ω *the structure vector.*

Extending these results to the entire quantionic algebra $\mathcal{O}_J$, the complex linear Minkowski space, $M_0(\mathbb{C}) \stackrel{\text{def}}{=} M_0 \oplus iM_0$, becomes the space of coefficients. In the next sections, we transfer the expressions for the product β from the algebra $\mathcal{O}_J$ to the coefficients $M_0(\mathbb{C})$. Since both the Lorentz frame and the structural frame are distinguished in the quantionic algebra, we shall express the product β in both.

3.6. The product β

Since the real quantions, $\mathcal{R}$, are defined by real 4-vectors, two real 4-vectors are needed to specify a general quantion $u \in \mathcal{O}_J$. We write

$$u = \omega^\tau U_\tau + J\sigma\omega^\tau U'_\tau\,, \tag{39}$$

where $U_\tau, U'_\tau \in M_0$. We collect these two real 4-vectors into one complex 4-vector,

$$u^\tau \stackrel{\text{def}}{=} U_\tau + \mathrm{i}U'_\tau \in M_0\,(\mathbb{C})\,. \tag{40}$$

The next theorem expresses the quantionic product β in tensorial form, the quantions being represented by complex 4-vectors. In this formalism, the machinery of tensor algebra is put to the service of algebraic computations. It also condenses into a single formula, (41) or (42), the multiplication tables for β, which were given by cases in the algebraic frames.

Theorem 12 *In covariant form, the product β reads*

$$(u\beta v)^\rho = (\Omega, v)\, u^\rho + (\Omega, u)\, v^\rho - (u, v)\, \Omega^\rho - \mathrm{i}\eta^{\rho\sigma}\varepsilon_{\sigma\alpha\beta\gamma}\Omega^\alpha u^\beta v^\gamma, \tag{41}$$

or, in coordinate-free notation,

$$u\beta v = (\Omega, v)\, u + (\Omega, u)\, v - (u, v)\, \Omega - i\, {}^*(\Omega \wedge u \wedge v)\,. \tag{42}$$

Proof. We write the product $w = u\beta v$, where $u, v, w \in \mathcal{O}_J$, in the expansion (39):

$$(\omega^\tau W_\tau + J\sigma\omega^\tau W'_\tau) = \left(\omega^\rho U_\rho + J\sigma\omega^\rho U'_\rho\right)\beta\,(\omega^\sigma V_\sigma + J\sigma\omega^\sigma V'_\sigma)\,. \tag{43}$$

To expand the right-hand side as a product of binomials,

$$r.h.s = (\omega^\rho\beta\omega^\sigma)\left(U_\rho V_\sigma - U'_\rho V'_\sigma\right) + J\sigma\,(\omega^\rho\beta\omega^\sigma)\left(U_\rho V'_\sigma + U'_\rho V_\sigma\right), \tag{44}$$

one needs the expressions for $\omega^\rho\beta\omega^\sigma$ and $J\sigma\,(\omega^\rho\beta\omega^\sigma)$,

$$\begin{aligned} \omega^\rho\beta\omega^\sigma &= \omega^\rho\sigma\omega^\sigma + J\sigma\,(\omega^\rho\alpha\omega^\sigma)\,, \\ J\sigma\,(\omega^\rho\beta\omega^\sigma) &= J\sigma\,(\omega^\rho\sigma\omega^\sigma) - \omega^\rho\alpha\omega^\sigma. \end{aligned}$$

By separating the real an imaginary parts in relation (43), one obtains

$$\omega^\tau w_\tau = (\omega^\rho\sigma\omega^\sigma)\left(U_\rho V_\sigma - U'_\rho V'_\sigma\right) - (\omega^\rho\alpha\omega^\sigma)\left(U_\rho V'_\sigma + U'_\rho V_\sigma\right), \tag{45}$$

$$\omega^\tau W_\tau = (\omega^\rho\alpha\omega^\sigma)\left(U_\rho V_\sigma - U'_\rho V'_\sigma\right) + (\omega^\rho\sigma\omega^\sigma)\left(U_\rho V'_\sigma + U'_\rho V_\sigma\right). \tag{46}$$

In the transition from the quantionic algebra to the tensorial algebra of coefficients isomorphic to it, the imaginary unit J is eliminated, and its role taken over by the ordinary imaginary unit i. Beginning with the substitutions

$$\omega^{\rho\sigma} \stackrel{\text{def}}{=} (\omega^\rho\sigma\omega^\sigma) + \mathrm{i}\varepsilon\,(\omega^\rho\alpha\omega^\sigma)\,, \tag{47}$$

where $\varepsilon = \pm 1$, relations (45) and (46) are compacted into a single complex equation

$$\omega^\tau w_\tau = \omega^{\rho\sigma} u_\rho v_\sigma\,. \tag{48}$$

The next task is to compute $\omega^{\rho\sigma}$ beginning with $\omega^{\rho}\sigma\omega^{\sigma}$ and $\omega^{\rho}\alpha\omega^{\sigma}$.

$$\begin{aligned}\omega^{\rho}\sigma\omega^{\sigma} &= \left(e\Omega^{\rho}+\frac{1}{2}\varepsilon^{\gamma\tau\mu\nu}\Omega_{\gamma}\,e_{\mu\nu}\right)\sigma\left(e\Omega^{\sigma}+\frac{1}{2}\varepsilon^{\delta\sigma\alpha\beta}\Omega_{\delta}\,e_{\alpha\beta}\right)\\ &= e\Omega^{\rho}\Omega^{\sigma}+\varepsilon\frac{1}{2}\varepsilon^{\gamma\tau\mu\nu}\Omega_{\gamma}\Omega^{\sigma}e_{\mu\nu}+\varepsilon\frac{1}{2}\varepsilon^{\delta\sigma\alpha\beta}\Omega_{\delta}\Omega^{\rho}e_{\alpha\beta}\\ &\quad+\frac{1}{4}\varepsilon^{\gamma\rho\mu\nu}\varepsilon^{\delta\sigma\alpha\beta}\Omega_{\delta}\Omega_{\gamma}\left(e_{\mu\nu}\,\sigma\,e_{\alpha\beta}\right).\end{aligned}$$

The last term, computed separately, is:

$$\begin{aligned}&\frac{1}{4}\varepsilon^{\gamma\rho\mu\nu}\varepsilon^{\delta\sigma\alpha\beta}\Omega_{\gamma}\Omega_{\delta}\varepsilon_{\mu\nu\alpha\beta}j-\frac{1}{4}\varepsilon^{\gamma\rho\mu\nu}\varepsilon^{\delta\sigma\alpha\beta}\Omega_{\gamma}\Omega_{\delta}\left(\eta_{\mu\alpha}\,\eta_{\nu\beta}-\eta_{\mu\beta}\,\eta_{\nu\alpha}\right)e\\ =\;&\left(\Omega^{\rho}\Omega^{\sigma}-\eta^{\rho\sigma}\right)e.\end{aligned}$$

Hence,

$$\omega^{\rho}\sigma\omega^{\sigma}=\left(2\Omega^{\rho}\Omega^{\sigma}-\eta^{\rho\sigma}\right)e+\frac{1}{2}\left(\varepsilon^{\gamma\rho\alpha\beta}\Omega^{\sigma}+\varepsilon^{\gamma\sigma\alpha\beta}\Omega^{\rho}\right)\Omega_{\gamma}e_{\alpha\beta}\,. \tag{49}$$

Next, we compute $\omega^{\rho}\alpha\omega^{\sigma}$:

$$\begin{aligned}\omega^{\rho}\alpha\omega^{\sigma} &= \frac{1}{4}\Omega_{\gamma}\Omega_{\delta}\varepsilon^{\gamma\rho\mu\nu}\varepsilon^{\delta\sigma\alpha\beta}\left(e_{\mu\nu}\,\alpha\,e_{\alpha\beta}\right)\\ &= \Omega_{\gamma}\Omega_{\delta}\varepsilon^{\gamma\rho\mu\nu}\varepsilon^{\delta\sigma\alpha\beta}\eta_{\mu\alpha}e_{\nu\beta}\\ &= \left(\eta^{\rho\beta}\eta^{\nu\sigma}-\eta^{\nu\sigma}\Omega^{\rho}\Omega^{\beta}+\eta^{\nu\rho}\Omega^{\sigma}\Omega^{\beta}\right)e_{\nu\beta}\,.\end{aligned} \tag{50}$$

Combining these partial results into the expression (47), one obtains

$$\begin{aligned}\omega^{\rho\sigma} &= \left(2\Omega^{\rho}\Omega^{\sigma}-\eta^{\rho\sigma}\right)e+\frac{1}{2}\varepsilon^{\gamma\tau\alpha\beta}\left(\Omega^{\sigma}\delta^{\rho}_{\tau}+\Omega^{\rho}\delta^{\sigma}_{\tau}\right)\Omega_{\gamma}e_{\alpha\beta}\\ &\quad+\varepsilon\mathrm{i}\left(\eta^{\rho\beta}\eta^{\alpha\sigma}-\eta^{\alpha\sigma}\Omega^{\rho}\Omega^{\beta}+\eta^{\alpha\rho}\Omega^{\sigma}\Omega^{\beta}\right)e_{\alpha\beta}\,.\end{aligned} \tag{51}$$

We now make the substitutions (37) for ω^{τ} and (51) for $\omega^{\rho\sigma}$ into relation (48), separating the e-component from the $e_{\alpha\beta}$-component:

$$\Omega^{\tau}w_{\tau}=\left(2\Omega^{\rho}\Omega^{\sigma}-\eta^{\rho\sigma}\right)u_{\rho}v_{\sigma}=2\left(\Omega,u\right)\left(\Omega,v\right)-\left(u,v\right). \tag{52}$$

$$\begin{aligned}\frac{1}{2}\varepsilon^{\sigma\tau\alpha\beta}\Omega_{\sigma}w_{\tau} &= \frac{1}{2}\varepsilon^{\gamma\tau\alpha\beta}\left(\Omega^{\sigma}\delta^{\rho}_{\tau}+\Omega^{\rho}\delta^{\sigma}_{\tau}\right)\Omega_{\gamma}u_{\rho}v_{\sigma}\\ &\quad+\mathrm{i}\varepsilon\left(\eta^{\rho\beta}\eta^{\alpha\sigma}-\eta^{\alpha\sigma}\Omega^{\rho}\Omega^{\beta}+\eta^{\alpha\rho}\Omega^{\sigma}\Omega^{\beta}\right)u_{\rho}v_{\sigma}\,.\end{aligned} \tag{53}$$

Contracting both sides by $2\varepsilon_{\kappa\lambda\alpha\beta}$ yields

$$\begin{aligned}(\delta^\sigma_\lambda\delta^\tau_\kappa-\delta^\sigma_\kappa\delta^\tau_\lambda)\,\Omega_\sigma w_\tau &= (\delta^\gamma_\lambda\delta^\tau_\kappa-\delta^\gamma_\kappa\delta^\tau_\lambda)\,(\Omega^\sigma\delta^\rho_\tau+\Omega^\rho\delta^\sigma_\tau)\,\Omega_\gamma u_\rho v_\sigma\\ &\quad+\varepsilon 2\mathrm{i}\varepsilon_{\kappa\lambda\alpha\beta}\left(\eta^{\rho\beta}\eta^{\alpha\sigma}-\eta^{\alpha\sigma}\Omega^\rho\Omega^\beta+\eta^{\alpha\rho}\Omega^\sigma\Omega^\beta\right)u_\rho v_\sigma .\end{aligned}$$

Hence

$$\begin{aligned}\Omega_\lambda w_\kappa-\Omega_\kappa w_\lambda &= (\Omega,v)\,u_\kappa\Omega_\lambda+(\Omega,u)\,v_\kappa\Omega_\lambda-(\Omega,v)\,u_\lambda\Omega_\kappa-(\Omega,u)\,v_\lambda\Omega_\kappa\\ &\quad+\varepsilon 2\mathrm{i}\varepsilon_{\kappa\lambda\alpha\beta}\left(\eta^{\rho\beta}\eta^{\alpha\sigma}-\eta^{\alpha\sigma}\Omega^\rho\Omega^\beta+\eta^{\alpha\rho}\Omega^\sigma\Omega^\beta\right)u_\rho v_\sigma\,.\end{aligned}$$

This relation can be separated into two simpler ones, specifically, the relation

$$\begin{aligned}\Omega_\lambda w_\kappa &= (\Omega,v)\,u_\kappa\Omega_\lambda+(\Omega,u)\,v_\kappa\Omega_\lambda\\ &\quad+\varepsilon\mathrm{i}\varepsilon_{\kappa\lambda\alpha\beta}\left(\eta^{\rho\beta}\eta^{\alpha\sigma}-\eta^{\alpha\sigma}\Omega^\rho\Omega^\beta+\eta^{\alpha\rho}\Omega^\sigma\Omega^\beta\right)u_\rho v_\sigma+S_{\lambda\kappa}\end{aligned}$$

and its symmetric counterpart obtained by interchanging κ and λ. The term $S_{\lambda\kappa}=S_{\kappa\lambda}$ is arbitrary but symmetric. Contracting both sides by Ω^λ, one obtains

$$w_\kappa=(\Omega,v)\,u_\kappa+(\Omega,u)\,v_\kappa+\varepsilon\mathrm{i}\varepsilon_{\kappa\lambda\alpha\beta}\Omega^\lambda u^\beta v^\alpha+\Omega^\lambda S_{\lambda\kappa}.$$

To determine the last arbitrary term, contracting by Ω^κ and comparing the result with relation (52) yields

$$(\Omega,w)=2\,(\Omega,v)\,(\Omega,u)+\Omega^\kappa\Omega^\lambda S_{\lambda\kappa}=2\,(\Omega,u)\,(\Omega,v)-(u,v)\ .$$

Hence, $\Omega^\lambda S_{\lambda\kappa}=-(u,v)\,\Omega_\kappa$. To write the final expression, (41), we select the orientation $\varepsilon=-1$ for a minor cosmetic advantage. ∎

We see that the 4-tensor Ω plays the role of the algebraic unit in (41), i.e., $\Omega\beta f\equiv f$.

The next theorem gives β in the structural frame, where we write $u=U\Omega+\vec{u}$. In this expression, U and $\vec{u}$ are the projections of the complex 4-vector u on Ω and Π respectively:

$$\mathbb{C}\qquad U=(\Omega,u)\,, \tag{54}$$

$$\Pi\qquad \vec{u}=u-(\Omega,u)\,. \tag{55}$$

Theorem 13 *In structural form, the product β reads*

$$(U\Omega+\vec{u})\,\beta\,(V\Omega+\vec{v})=(UV+\vec{u}\cdot\vec{v})\,\Omega+U\vec{v}+\vec{u}V+i\vec{u}\times\vec{v}\,. \tag{56}$$

Proof. The substitutions $u=U\Omega+\vec{u}$ and $v=V\Omega+\vec{v}$ into the formula (42) immediately yield this result. ∎

The next theorem serves two purposes. By proving the associativity of the product β directly, it verifies the integrity of the algebraic calculations that led to the expression (41), but, more importantly, it shows that associativity is not linked to there existing a unit in the quantionic algebra. A structure vector Ω is needed, but this vector need not have the interpretation of an algebraic unit.

Theorem 14 *The product β, defined by the relation (41), is associative for all vectors Ω^ρ, i.e., time-like, null, or space-like, and of any magnitude,*

$$(u\beta v)\,\beta w = u\beta\,(v\beta w)\,. \tag{57}$$

Proof. Let us write the product β as a sum of a symmetric and an antisymmetric part,

$$u\beta v = ru\beta_1 v + su\beta_2 v\,, \tag{58}$$

where, by relation (41),

$$u\beta_1 v = (v,\Omega)\,u^\rho + (u,\Omega)\,v^\rho - (u,v)\,\Omega^\rho,$$

$$u\beta_2 v = \varepsilon^\rho{}_{\alpha\beta\gamma}\Omega^\alpha u^\beta v^\gamma.$$

Even though they are known, we treat the coefficients r and s as arbitrary. Substitution of relation (58) into the associator yields

$$[u,v,w] = r^2\,[u,v,w]_1 + s^2\,[u,v,w]_2 + rs\,[u,v,w]_{12}\,,$$

where

$$[u,v,w]_{12} \stackrel{\mathrm{def}}{=} (u\beta_1 v)\,\beta_2 w - u\beta_1\,(v\beta_2 w) + (u\beta_2 v)\,\beta_1 w - u\beta_2\,(v\beta_1 w)\,.$$

We first show that this term vanishes (as terms of the type $\varepsilon^\rho_{\alpha\beta\gamma}\Omega^\alpha\Omega^\beta$ vanish by themselves, we need not write them).

$$\begin{aligned} [u,v,w]^\rho_{12} &= \varepsilon^\rho_{\alpha\beta\gamma}\Omega^\alpha\left[(u,\Omega)\,v^\beta + (v,\Omega)\,u^\beta\right]w^\gamma \\ &\quad -\varepsilon^\rho_{\alpha\beta\gamma}\Omega^\alpha u^\beta\left[(v,\Omega)\,w^\gamma + (w,\Omega)\,v^\gamma\right] \\ &\quad + (\Omega,w)\,\varepsilon^\rho_{\alpha\beta\gamma}\Omega^\alpha u^\beta v^\gamma - \varepsilon_{\lambda\alpha\beta\gamma}\Omega^\alpha u^\beta v^\gamma w^\lambda\Omega^\rho \\ &\quad - (\Omega,u)\,\varepsilon^\rho_{\alpha\beta\gamma}\Omega^\alpha v^\beta w^\gamma + \varepsilon_{\lambda\alpha\beta\gamma}\Omega^\alpha v^\beta w^\gamma u^\lambda\Omega^\rho \\ &= 0\,. \end{aligned}$$

Next we compute $[u,v,w]_1$:

$$
\begin{aligned}
& [u,v,w]_1^\rho \\
= \;& (u,\Omega)\,(v,\Omega)\,w^\rho + (v,\Omega)\,(u,\Omega)\,w^\rho - \varpi\,(u,v)\,w^\rho \\
& + (w,\Omega)\,(u,\Omega)\,v^\rho + (w,\Omega)\,(v,\Omega)\,u^\rho - (w,\Omega)\,(u,v)\,\Omega^\rho \\
& - (u,\Omega)\,(v,w)\,\Omega^\rho - (v,\Omega)\,(u,w)\,\Omega^\rho + (u,v)\,(w,\Omega)\,\Omega^\rho \\
& - (v,\Omega)\,(w,\Omega)\,u^\rho - (w,\Omega)\,(v,\Omega)\,u^\rho + \varpi\,(v,w)\,u^\rho \\
& - (u,\Omega)\,(v,\Omega)\,w^\rho - (u,\Omega)\,(w,\Omega)\,v^\rho + (u,\Omega)\,(v,w)\,\Omega^\rho \\
& + (v,\Omega)\,(u,w)\,\Omega^\rho + (w,\Omega)\,(u,v)\,\Omega^\rho - (v,w)\,(u,\Omega)\,\Omega^\rho \\
= \;& \varpi\,[(v,w)\,u^\rho - (u,v)\,w^\rho] + (v,\Omega)\,[(u,\Omega)\,w^\rho - (w,\Omega)\,u^\rho] \\
& + [(w,\Omega)\,(u,v) - (u,\Omega)\,(v,w)]\,\Omega^\rho\,.
\end{aligned}
$$

The computation of $[u,v,w]_2$ yields

$$
\begin{aligned}
& [u,v,w]_2 \\
= \;& \varepsilon^\rho{}_{\lambda\mu\nu}\varepsilon^\mu{}_{\alpha\beta\gamma}\Omega^\lambda\Omega^\alpha u^\beta v^\gamma w^\nu - \varepsilon^\rho{}_{\lambda\nu\mu}\varepsilon^\mu{}_{\alpha\beta\gamma}\Omega^\lambda\Omega^\alpha u^\nu v^\beta w^\gamma \\
= \;& \left[\varepsilon^\rho{}_{\lambda\mu\nu}\varepsilon^\mu{}_{\alpha\beta\gamma} - \varepsilon^\rho{}_{\lambda\mu\beta}\varepsilon^\mu{}_{\alpha\nu\gamma}\right]\Omega^\lambda\Omega^\alpha u^\beta v^\gamma w^\nu \\
= \;& [-\delta^\rho_\alpha\,\eta_{\beta\lambda}\,\eta_{\gamma\nu} - \delta^\rho_\gamma\eta_{\alpha\lambda}\,\eta_{\beta\nu} - \delta^\rho_\beta\,\eta_{\gamma\lambda}\,\eta_{\alpha\nu} + \delta^\rho_\alpha\,\eta_{\beta\nu}\,\eta_{\gamma\lambda} + \delta^\rho_\gamma\,\eta_{\alpha\nu}\,\eta_{\beta\lambda} \\
& + \delta^\rho_\beta\,\eta_{\gamma\nu}\,\eta_{\alpha\lambda} + \delta^\rho_\alpha\,\eta_{\lambda\nu}\,\eta_{\gamma\beta} + \delta^\rho_\gamma\eta_{\alpha\lambda}\eta_{\beta\nu} + \delta^\rho_\nu\,\eta_{\gamma\lambda}\,\eta_{\alpha\beta} - \delta^\rho_\alpha\,\eta_{\beta\nu}\,\eta_{\gamma\lambda} \\
& - \delta^\rho_\gamma\,\eta_{\alpha\beta}\,\eta_{\nu\lambda} - \delta^\rho_\nu\,\eta_{\gamma\beta}\,\eta_{\alpha\lambda}]\Omega^\lambda\Omega^\alpha u^\beta v^\gamma w^\nu \\
= \;& \varpi\,[(v,w)\,u^\rho - (u,v)\,w^\rho] + (v,\Omega)\,[(u,\Omega)\,w^\rho - (w,\Omega)\,u^\rho] \\
& + [(w,\Omega)\,(u,v) - (u,\Omega)\,(v,w)]\,\Omega^\rho.
\end{aligned}
$$

Hence,

$$[u,v,w]_1 = [u,v,w]_2\,,$$

and, consequently

$$[u,v,w] = \left(r^2 + s^2\right)[u,v,w]_1\,.$$

Since the vectors $u^\rho, v^\rho, w^\rho \in \Pi$ are arbitrary, the associator $[u,v,w]_1$ does not vanish identically, which implies that the product beta defined by relation (58) is associative if and only if $r^2 + s^2 = 0$, which is the case since $r = 1$ and $s = \pm\mathrm{i}$. We also conclude that the product β is associative for all three types of structure vectors (time-like, space-like or null), which may also be of any length. ■

This completes the extraction of the quantionic algebra embedded in the only non-trivial physical non-unitary quantal algebra. In future work, we shall denote this algebra by the symbol $\mathbb{D}$, instead of $\mathcal{O}_J$.

References

[1] Emile Grgin, *Relativistic Ring Extension of the Field of Complex Numbers*, Phys. Letters B **431** (1998) 15.

[2] Emile Grgin, *Inherently Relativistic Quantum Theory. Part I. The Algebra of Observables*, Fizika B (Zagreb) **10**, 3 (2001) 113.

[3] Emile Grgin, *Inherently Relativistic Quantum Theory. Part II. Classification of the Solutions*, Fizika B (Zagreb) **10**, 3 (2001) 139.

[4] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vols. I and II, Cambridge University Press, Cambridge (1986).

[5] R. S. Ward and R. O Wells, Jr., *Twistor Geometry and Field Theory*, Cambridge University Press, Cambridge (1990).

SUŠTINSKI RELATIVISTIČKA KVANTNA TEORIJA
III Dio. KVANTIONSKA ALGEBRA

Kvantna mehanika i relativnost nisu uskladive na strukturnoj razini, što vrlo otežava njihovo ujedinjenje. Neuskladivost može značiti da potpuna kvantna teorija ujedinjena s relativnošću postoji, ali nije poznata, dok standardna kvantna mehanika, kao poseban slučaj, ne može biti relativistička. Ako je tako, traženje poopćenja je opravdano, no pitanje je kako. Stara je zamisao zamijeniti polje kompleksnih brojeva strukturno bogatijom algebrom, ali do sada takvi pokušaji nisu doveli kvantnu teoriju bliže relativnosti. Ovaj je rad također zasnovan na toj zamisli, ali, za razliku od ranijih pokušaja, ne traži nov brojevni sustav u postojećim matematičkim strukturama. Na osnovi općih razmatranja razvijenim u prva dva dijela ovog rada, izvedena je nova matematička struktura, nazvana *kvantionska algebra*, kao teorem u ovom članku. Ona je jedinstvena, u postavci relativistička i poopćuje polje kompleksnih brojeva skladno kvantnoj teoriji.