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SOLUTIONS OF THE CHIRAL DIRAC EQUATION

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Massive, chirally right-handed solutions of an antilinearly modified Dirac equation are calculated. This is a continuation of a previous paper, where the equation was first introduced and found to separate into states of definite (and opposite) chiralities, each state being either massive or massless.

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1. *Introduction*

In previous papers, the Dirac equation with two mass parameters and related topics were discussed (see Refs. [1] through [6]). The approach was used to derive standard equations for massive, massless and tachyonic fermions. In particular, a massless equation was obtained, which differs from the usual one and does not produce a superfluous conserved current. In Ref. [7], the aforementioned results were reformulated and justified on the grounds of desirable features relating to the active symmetry operations (time reversal, spatial parity, etc.). Possible applications and a flavored neutrino model were examined in Refs. [1], [3], [7] and [8].

In Ref. [9], an antilinear modification was introduced, which generates a formalism of two chiral eigenstates, each being either massive or massless. This description is quite different from that of the other references, but, again, could be useful for neutrino physics, as pointed out in the Conclusions of Ref. [9]. For related approaches, see Refs. [10] through [13].

In this paper, massive right-handed solutions of the "chiral Dirac equation" of Ref. [9] are calculated. The elementary solutions are not energy eigenstates, but consist of linear combinations of positive and negative energy states; more general solutions are obtained by superposition. The treatment is done before second quantization, but issues of second quantization are discussed in Sects. 2, 4 and 5.

Notation is rather conventional: specifically, and unless otherwise noted, Greek (Latin) indices run through the values $0, 1, 2, 3$ $(1, 2, 3)$ and the summation convention is applied to repeated up and down labels. Units are such that $\hbar = c = 1$. An attempt is made at distinguishing powers from superscripts: for instance, $(\not{\mathcal{P}})^2$ and $|\,a\,|^2$ are powers, while γ^2 indicates a specific object with superscript 2. The curly bracket is used for ordered sets: e.g., $\{x^\omega\}$ denotes four objects in the order $0-3$.

2. Review

In a frame of reference $\mathcal{X}$ of real spacetime coordinates $x = \{x^\omega\}$ and pseudoeuclidean metric $g^{\mu\nu} = \mathrm{diag}\{+1, -1, -1, -1\}$, the chiral Dirac equation introduced in Ref. [9] may be written as follows

$$\not{\mathcal{P}}\Psi(x) = \mathrm{M}(a, b)\,[\Psi(x)]^{\mathsf{C}}, \tag{1}$$

with

$$\not{\mathcal{P}} = \mathrm{i}\gamma^\alpha \partial_\alpha \tag{2}$$

and

$$\mathrm{M}(a, b) = a\,\mathrm{M}^{(-)} + \,b\,\mathrm{M}^{(+)}\,, \tag{3}$$

where a and b are complex constants, $\Psi(x)$ is a complex four-spinor, and $\mathrm{M}^{(\mp)}$ indicate the chiral projectors:

$$\mathrm{M}^{(\mp)} = \frac{1}{2}\,(I \mp \varepsilon\gamma^5)\,. \tag{4}$$

The Dirac matrices γ^ω (in a fixed chosen representation) obey the usual rules

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I\,, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,, \tag{5}$$

with I being the 4×4 identity matrix. The matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ is hermitian and unitary, and anticommutes with all γ^ω. For general reference on the Dirac equation and related topics, see, for instance Refs. [14] through [24]. The value of the sign $\varepsilon = (-1)^{T+S}$ depends on the frame of reference [1, 25]. Namely, the time-index T and the space-index S of $\mathcal{X}$ are so defined: $T = 0$ if $t = x^0$ runs forward ($T = 1$ otherwise) and $S = 0$ if $s = \{x^\ell\}$ is a right-handed triplet ($S = 1$ otherwise). It is also reminded that

$$(\gamma^\mu)^* = \mathrm{B}^\dagger\gamma^\mu\mathrm{B}\,, \qquad (\gamma^5)^* = -\mathrm{B}^\dagger\gamma^5\mathrm{B}\,, \tag{6}$$

where B is the (fixed chosen) unitary matrix [2] associated with charge conjugation, and the asterisk denotes complex conjugation. The symbol $\mathtt{C}$ in Eq. (1) indicates the antilinear operation of charge conjugation, defined as [7]

$$[\Upsilon(x)]^{\mathtt{C}} = \gamma^5 \mathrm{B}\Upsilon^*(x) \tag{7}$$

on a generic four-spinor $\Upsilon(x)$; the notation $\Upsilon^{\mathtt{C}}(x)$ will also be used.

Equation (1) is manifestly covariant under changes of coordinates of the Poincaré group, provided a and b are treated as scalars, and the usual (passive) spinor transformations [20] are adopted with an appropriate phase convention (see Eq. (37) of Ref. [2]). For the active operations [2] of charge conjugation, spatial parity ($\mathtt{P}$), time reversal ($\mathtt{T}$), $\mathtt{PC}$ and $\mathtt{TPC}$

$$\Upsilon^{\mathtt{P}}(x) = \mathrm{i}\gamma^0\, \Upsilon(t,-s)\,, \qquad \Upsilon^{\mathtt{T}}(x) = \gamma^0 \mathrm{B}\Upsilon^*(-t,s)\,, \tag{8}$$

$$\Upsilon^{\mathtt{PC}}(x) = \mathrm{i}\gamma^0\gamma^5 \mathrm{B}\Upsilon^*(t,-s)\,, \qquad \Upsilon^{\mathtt{TPC}}(x) = -\mathrm{i}\gamma^5\, \Upsilon(-x)\,, \tag{9}$$

one obtains that: (i) $\mathtt{TPC}$ invariance is valid for all possible choices of a and b; (ii) invariance under $\mathtt{C}$ applies if $b = a^*$; (iii) invariance under $\mathtt{P}$ is valid if $a = b$; (iv) invariance under $\mathtt{T}$ applies if $a, b \in \Re$; (v) invariance under $\mathtt{PC}$ is valid if $a, b \in \Re$. For example, the equations for $\Psi^{\mathtt{C}}(x)$ and $\Psi^{\mathtt{T}}(x)$ are as follows

$$\not{\mathcal{P}}\Psi^{\mathtt{C}}(x) = \mathrm{M}(b^*, a^*)\, [\Psi^{\mathtt{C}}(x)]^{\mathtt{C}}, \tag{10}$$

$$\not{\mathcal{P}}\Psi^{\mathtt{T}}(x) = \mathrm{M}(a^*, b^*)\, [\Psi^{\mathtt{T}}(x)]^{\mathtt{C}}. \tag{11}$$

It is noted that Eq. (1) leads to the generalized (linear) Klein-Gordon equation

$$\Box\Psi(x) = \mathrm{N}(a,b)\, \Psi(x)\,, \tag{12}$$

with

$$\mathrm{N}(a,b) = |\,b\,|^2\, \mathrm{M}^{(-)} + \,|\,a\,|^2\, \mathrm{M}^{(+)} = \,\mathrm{M}\left(|\,b\,|^2, |\,a\,|^2\right), \tag{13}$$

and

$$\Box = -\,\partial_\alpha g^{\alpha\beta}\partial_\beta = (\not{\mathcal{P}})^2. \tag{14}$$

Furthermore, it is observed that Eq. (1) is nearly linear, but not exactly linear; specifically, if $\Psi_1(x)$ and $\Psi_2(x)$ are solutions of (1), their linear combination may not be a solution of (1) unless the coefficients of the combination are real. For lack of a better term, this property will be called "$\Re$-linearity".

Using definitions

$$\Lambda(x) = \mathrm{M}^{(-)}\, \Psi(x)\,, \qquad \Phi(x) = \mathrm{M}^{(+)}\, \Psi(x)\,, \tag{15}$$

Eqs. (1) and (12) split into left-handed (LH) and right-handed (RH) equations:

$$\not{\partial}\Lambda(x) = b\,[\Lambda(x)]^{\mathsf{C}}\,, \qquad \Box\Lambda(x) = |\,b\,|^2\,\Lambda(x)\,, \tag{16}$$

$$\not{\partial}\Phi(x) = a\,[\Phi(x)]^{\mathsf{C}}\,, \qquad \Box\Phi(x) = |\,a\,|^2\,\Phi(x)\,, \tag{17}$$

with separately conserved real currents

$$j^{\mu}(x) = \overline{\Lambda}(x)\gamma^{\mu}\Lambda(x)\,, \tag{18}$$

$$k^{\mu}(x) = \overline{\Phi}(x)\gamma^{\mu}\Phi(x)\,, \tag{19}$$

where

$$\overline{\Upsilon}(x) = \Upsilon^{\dagger}(x)\gamma^{0}\,. \tag{20}$$

Each current is appropriate for the usual probability interpretation in terms of a single particle theory [7, 20]. Thus, Eq. (1) describes two Dirac particle states, each with a definite chirality. If $b{\neq}0$ (if $a{\neq}0$), the spinor Λ (the spinor Φ) is massive; otherwise, it is massless. Tachyonic states are not allowed. Note: in the second quantization [19], the current for a massive chiral state does not remain conserved if quantized with anticommutators (see below).

Remark. In Ref. [9], the case $b = 0, a{\neq}0$ was prospected. The (linear) LH massless equation

$$\not{\partial}\Lambda(x) = 0 \tag{21}$$

is well known; the massive RH equation

$$\not{\partial}\Phi(x) = a\,[\Phi(x)]^{\mathsf{C}}\,, \qquad a{\neq}0 \tag{22}$$

is $\Re$-linear, and its current (19) satisfies

$$\partial_{\alpha}k^{\alpha}(x) = -\mathrm{i}a\overline{\Phi}(x)\Phi^{\mathsf{C}}(x) - (\mathrm{i}a)^{*}\,[\Phi^{\mathsf{C}}(x)]^{\dagger}\gamma^{0}\Phi(x)\,. \tag{23}$$

Due to the antisymmetry of the matrix $\gamma^0\gamma^5\,\mathrm{B}$, each of the two terms at the right-hand-side of Eq. (23) vanishes identically, provided the spinor components of $\Phi(x)$ are treated as commuting quantities (which is certainly the case before second quantization is applied). Specifically, the first term at the right-hand-side of Eq. (23) can be expressed as follows:

$$-\mathrm{i}a\overline{\Phi}(x)\Phi^{\mathsf{C}}(x) = -\mathrm{i}a\Phi^{\dagger}(x)\,\gamma^{0}\gamma^{5}\,\mathrm{B}\,[\Phi^{\dagger}(x)]^{\sharp}\,, \tag{24}$$

where $\sharp$ indicates transposition; the second term is similar. In a second quantization scheme with anticommutators [19], the right-hand-side of Eq. (23) does not vanish identically; thus, the current is not generally conserved. For this reason, Eq. (22) does not appear to be a particularly obvious candidate for a second quantization procedure of the standard fermionic type.

3. Elementary solutions

It is readily verified that Eq. (22) admits simple solutions of the form

$$\Phi(x;p) = \mathit{\Phi}(p)\exp(-\mathrm{i}p_\alpha x^\alpha) - \frac{a}{|\,a\,|^2}\, p_\beta \gamma^\beta \gamma^5 \mathrm{B}\, \mathit{\Phi}^*(p)\exp(\mathrm{i}p_\alpha x^\alpha)\,, \tag{25}$$

where $p = \{p_\omega\}$ are real parameters constrained by the mass-shell condition

$$p_\alpha g^{\alpha\beta} p_\beta = |\,a\,|^2\,. \tag{26}$$

The symbol $\mathit{\Phi}(p)$ denotes an arbitrary complex RH four-spinor, independent of spacetime. This may be written as

$$\mathit{\Phi}(p) = d_h(p)\,\mathit{\Phi}^h \qquad (h = 1, 2), \tag{27}$$

if two constant (and linearly independent) RH four-spinors $\mathit{\Phi}^h$ are introduced, and are linearly combined by means of the arbitrary complex coefficients $d_h(p)$.

It is convenient to resolve the quadratic constraint (26) as follows:

$$p_0 = (-1)^T \eta\, E(\mathbf{p})\,, \qquad \eta = \pm 1, \tag{28}$$

having defined the "unsigned" energy

$$E(\mathbf{p}) = \left[\,|\,a\,|^2 + (p_1)^2 + (p_2)^2 + (p_3)^2\right]^{1/2} \,>\, 0\,, \tag{29}$$

with $\mathbf{p} = \{p_\ell\}$. Equation (28) specifies p_0 as a "positive energy" or as a "negative energy" [7], depending on whether the sign η is positive or negative. The solutions (25) corresponding to these two values of η are indicated below:

$$\begin{aligned}\Phi^{[\eta]}(x;\mathbf{p}) = {} & d_h^{[\eta]}(\mathbf{p})\,\mathit{\Phi}^h \exp\Big(-\mathrm{i}p_\alpha^{[\eta]} x^\alpha\Big) \\ & - \frac{a}{|\,a\,|^2}\, p_\beta^{[\eta]} \gamma^\beta \gamma^5 \mathrm{B}\left[d_h^{[\eta]}(\mathbf{p})\,\mathit{\Phi}^h\right]^* \exp\Big(\mathrm{i}p_\alpha^{[\eta]} x^\alpha\Big)\,,\end{aligned} \tag{30}$$

where

$$p_\ell^{[\eta]} = p_\ell\,, \qquad p_0^{[\eta]} = (-1)^T \eta\, E(\mathbf{p})\,, \tag{31}$$

and the rest of the notation is straightforward.

4. Superpositions

Solutions that are more general than (30) are obtained by means of the superposition

$$\Phi(x) = \sum_{\eta=\pm 1} \int \Phi^{[\eta]}(x;\mathbf{p})\,\mathrm{d}\mathbf{p}\,, \tag{32}$$

where

$$\mathrm{d}\mathbf{p} = \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3\,, \tag{33}$$

with the integration rule

$$-\infty < p_\ell < +\infty\,. \tag{34}$$

After some algebra, Eq. (32) may be rewritten as

$$\Phi(x) = W_h(x)\, \mathit{\Phi}^h + \frac{a}{|\,a\,|^2}\, \gamma^\beta\, W_{\beta h}(x)\, \left[\mathit{\Phi}^h\right]^{\mathsf{C}}, \tag{35}$$

with the definitions

$$W_h(x) = \int \left[f_h(\mathbf{q}) \exp(-\mathrm{i}q_\alpha x^\alpha) + g_h^*(\mathbf{q}) \exp(\mathrm{i}q_\alpha x^\alpha)\right] \mathrm{d}\mathbf{q}\,, \tag{36}$$

$$\begin{aligned} W_{\beta h}(x) = &\int q_\beta \left[g_h(\mathbf{q}) \exp(-\mathrm{i}q_\alpha x^\alpha) - f_h^*(\mathbf{q}) \exp(\mathrm{i}q_\alpha x^\alpha)\right] \mathrm{d}\mathbf{q} \\ &= \mathrm{i}\partial_\beta W_h^*(x)\,, \end{aligned} \tag{37}$$

where

$$f_h(\mathbf{p}) = d_h^{[+1]}(\mathbf{p})\,, \qquad g_h(\mathbf{p}) = \left[\, d_h^{[-1]}(-\mathbf{p})\right]^*\,, \tag{38}$$

and

$$q = p^{\,[+1]}\,. \tag{39}$$

In a more compact notation, Eqs. (30) and (35) can be reformulated as follows

$$\Phi^{[\eta]}(x;\mathbf{p}) = \mathcal{D}\left[d_h^{[\eta]}(\mathbf{p})\, \mathit{\Phi}^h \exp\Big(-\mathrm{i}p_\alpha^{[\eta]} x^\alpha\Big)\right], \tag{40}$$

and

$$\Phi(x) = \mathcal{D}\left[W_h(x)\, \mathit{\Phi}^h\right], \tag{41}$$

where $\mathcal{D}$ indicates the operator

$$\mathcal{D} = I + \frac{a}{|\,a\,|^2} \not{\mathcal{P}} \mathcal{C}\,, \tag{42}$$

and $\mathcal{C}$ is the charge conjugation operator

$$\mathcal{C}\Upsilon(x) = [\Upsilon(x)]^{\mathsf{C}}\,. \tag{43}$$

Remark. In a given frame of reference, it is always possible to choose the RH spinors Φ^h such that

$$\gamma^0 \left[\Phi^1\right]^{\mathsf{C}} = \Phi^2 \,, \qquad \gamma^0 \left[\Phi^2\right]^{\mathsf{C}} = -\Phi^1 \,, \tag{44}$$

$$\left[\Phi^h\right]^{\dagger} \Phi^k = \delta^{hk} \qquad (h,k = 1,2)\,, \tag{45}$$

where δ^{hk} is the Kronecker delta symbol. This will be useful in the following section. It is also noted that the d, f and g coefficients (and their complex conjugates) would be promoted to operators in second quantization [19], while the remaining quantities in the expressions (30) and (35) would be unaffected.

5. *Conclusions*

The spinorial structure of the elementary solutions (30) is best exposed in a time-forward frame of reference ($T = 0$), and with $p_\ell = 0$. In that case, using spinors Φ^h which satisfy Eq. (44), one obtains:

$$\begin{aligned}\Phi^{[+1]}(x;\mathbf{0}) = &\left[f_1(\mathbf{0})\exp(-\mathrm{i}\,|a|\,t) + \frac{a}{|\,a\,|}\, f_2^*(\mathbf{0})\exp(\mathrm{i}\,|a|\,t)\right]\Phi^1 \\ &+ \left[f_2(\mathbf{0})\exp(-\mathrm{i}\,|a|\,t) - \frac{a}{|\,a\,|}\, f_1^*(\mathbf{0})\exp(\mathrm{i}\,|a|\,t)\right]\Phi^2,\end{aligned} \tag{46}$$

and

$$\begin{aligned}\Phi^{[-1]}(x;\mathbf{0}) = &\left[g_1^*(\mathbf{0})\exp(\mathrm{i}\,|a|\,t) - \frac{a}{|\,a\,|}\, g_2(\mathbf{0})\exp(-\mathrm{i}\,|a|\,t)\right]\Phi^1 \\ &+ \left[g_2^*(\mathbf{0})\exp(\mathrm{i}\,|a|\,t) + \frac{a}{|\,a\,|}\, g_1(\mathbf{0})\exp(-\mathrm{i}\,|a|\,t)\right]\Phi^2.\end{aligned} \tag{47}$$

The number of internal degrees of freedom (counted from the number of f and g coefficients) totals four, that is, one degree of freedom for each pair (η, h). For a particle with a prescribed chirality, this number exceeds by a factor of two the number of degrees of freedom expected in a more conventional Dirac theory. As for the possibilities of second quantization [19], a bosonic scheme seems more feasible than a fermionic one, but neither is obvious at this point.

If the particle described by Eq. (22) is a chirally RH, massive, sterile (and possibly bosonic) partner [9] of the traditional neutrino of Eq. (21), the generalization of the formalism to curved spacetime should be considered, in order to examine gravitational effects and establish whether Eq. (22) might be suitable for the representation of dark matter [9, 26]. It is hoped that some clarifications on this can be obtained in future work; the interplay of helicity and chirality may also be investigated.

References

[1] A. Raspini, Fizika B **5** (1996) 159.
[2] A. Raspini, Fizika B **6** (1997) 123.
[3] A. Raspini, Fizika B **7** (1998) 83.
[4] A. Raspini, in *Photon and Poincaré Group*, Nova Science Publishers, Commack, New York (1999).
[5] Z. Tokuoka, Progr. Theor. Phys. **37** (1967) 603.
[6] N. D. S. Gupta, Nucl. Phys. B **4** (1967) 147.
[7] A. Raspini, Fizika B **7** (1998) 223.
[8] A. Raspini, Fizika B **8** (1999) 483.
[9] A. Raspini, Electromagnetic Phenomena **3** (2003) 46.
[10] G. Ziino, Annales Fondation Louis De Broglie **14** (1989) 427.
[11] A. O. Barut and G. Ziino, Mod. Phys. Lett. A **8** (1993) 1011.
[12] V. V. Dvoeglazov, Investigacion Cientifica **2** (2000) 59.
[13] V. V. Dvoeglazov, Annales Fondation Louis De Broglie **25** (2000) 81.
[14] I. J. R. Aitchison, *An Informal Introduction to Gauge Field Theories*, Cambridge University Press (1982).
[15] I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics*, IOP, Bristol (1989).
[16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York (1964).
[17] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience, New York (1959).
[18] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York (1980).
[19] M. Kaku, *Quantum Field Theory* (Oxford University Press, 1994).
[20] A. Messiah, *Quantum Mechanics*, Vol. II, John Wiley, New York (1966).
[21] J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Reading, MA (1973).
[22] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Peterson, Evanston, IL (1961).
[23] S. Weinberg, *The Quantum Theory of Fields*, Vol. I, Cambridge University Press (1995).
[24] F. Scheck, *Electroweak and Strong Interactions*, Springer-Verlag, Berlin (1996).
[25] A. Raspini, Int. J. Theor. Phys. **33** (1994) 1503.
[26] P. J. E. Peebles, in *Encyclopedia of Physics*, VCH Publishers, New York (1991).

RJEŠENJA KIRALNE DIRACOVE JEDNADŽBE

Izračunala sam masena, desno-kiralna rješenja antilinearno izmijenjene Diracove jednadžbe. Ovo je nastavak ranijeg članka, u kojemu se takva jednadžba prvi puta uvodi i gdje se našlo da se razdvaja u posebna stanja određene (suprotne) kiralnosti, a svako stanje je ili maseno ili bezmaseno.