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MESON EXCHANGE FORMALISM AND THE DEFINITION OF DELTA FUNCTIONS

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It is shown that in the simple context of the elementary one-meson exchanges, the use of the "improper" delta "functions" could lead to the physically correct results. The same results were obtained by involving limiting values of the integrals over "proper" functions, thus providing the examples of the sequences connected with delta "functions". That formulation of sequences of integrals emerged automatically from the physical considerations concerning hypernuclear processes.

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1. Introduction

The elementary calculation of an amplitude corresponding to a simple meson exchange process can illustrate and confirm a general use of "delta function" [1]. Such functions have appeared in theoretical physics in the 19th century [2]. Useful, compact and self consistent formalism has been developed by Dirac who said: *"Strictly, of course, $\delta(x)$ is not a proper function of x, but can be regarded only as a limit of a certain sequence of functions. All the same one can see $\delta(x)$ as though it were a proper function for practically all the purposes of quantum mechanics without getting incorrect results"* [3]. This improper function has been rigorously defined as sequence which in the limit defines the distribution delta [2,4,5]. The calculation of the meson exchanged amplitudes is easily and simply carried out in

[2]Deceased

the complex momentum transfer plane leading to the well known Yukawa-function radial dependence [6–8]. If the baryon masses are not equal, one immediately obtains a shifted Yukawa function [9], as encountered in the study of the hypernuclear decay. When the baryon masses are equal, the distributions appear in the direct calculation and no integration in the complex plane is needed. The unequal-mass case leads in the equal-mass-case limit to the same result "not involving improper functions" [11], i.e. when one deals with the strong nucleon–pion interaction. In the equal-mass case, one can also find the sequence, whose form is based on physical considerations and intuition. In the limit, that sequence leads to the same result.

2. *One-meson exchange*

A transition amplitude $A_{f\Lambda}$ corresponding to the one-pion (or any meson) exchange has a standard textbook generic form [6–10]. This form corresponds to either strong or weak meson exchange [9]. In the most general case displayed below

$$A_{f\Lambda} = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \int \mathrm{d}^4k\,\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-\mu^2+\mathrm{i}\epsilon}\,\langle\, f\,|\,T\big(S(x)W(y)\big)\,|\,\Lambda\,\rangle, \tag{1a}$$

with

$$k^2 = k_0^2 - \vec{k}^2,$$

$$kx = k_0x_0 - \vec{k}\cdot\vec{x}\,. \tag{1b}$$

Here $S(x)$ and $W(x)$ are some baryon densities (scalar or pseudoscalar, see Eq. (5a) below) which are sources of the meson (pion) field, with mass μ. The states $|\,f\,\rangle$ and $|\,\Lambda\,\rangle$ are some baryonic bound states. If made out of nucleons, like state $|\,f\,\rangle$, they symbolize atomic nuclei. Detailed form of those quantities need not concern us here. If all baryons are nucleons N, then the calculation can be found, for example, in Ref. [8], p. 213. In a more general case, the density $W(x)$ can contain a strange baryon, for example Λ hyperon [9], and one of the sources will be weak, meaning that neither parity nor strangeness need not be conserved. The details of their spatial properties and their precise particle content are immaterial for our arguments. The integration over $\mathrm{d}x_0$, $\mathrm{d}y_0$ and d^4k can be carried out without referring to these properties of the baryon densities. The results of such integrations do reveal the functional (distributional) properties mentioned in Introduction.

The time-ordered product in Eq. (1a) can be written as

$$\langle\, f\,|\,T\big(S(x)W(y)\big)\,|\,\Lambda\,\rangle = \theta(x_0-y_0)\sum_n \langle\, f\,|\,S(x)\,|\,n\,\rangle\langle\, n\,|\,W(y)\,|\,\Lambda\,\rangle$$

$$+\theta(y_0-x_0)\sum_s \langle\, f\,|\,W(y)\,|\,s\,\rangle\langle\, s\,|\,S(x)\,|\,\Lambda\,\rangle\;. \tag{2a}$$

Here the complete set of states $\sum_i |\,i\,\rangle\langle\,i\,| = 1$ was introduced [7]. If the weak interactions are involved [9], some of the nuclear states will describe hypernuclei. The density S (i.e. strong) conserves the strangeness while the density W (weak) changes the strangeness. In each matrix element in Eq. (2), as for example in

$$\langle\, f\,|\,S(x)\,|\,n\,\rangle\,, \tag{2b}$$

$$\langle\, f\,|\,W(y)\,|\,s\,\rangle\,, \tag{2c}$$

the strangeness is conserved [8–10]. Thus the intermediate states

$$|\,n\,\rangle; \qquad S = 0 \tag{3}$$

have no strangeness, while the states

$$|\,s\,\rangle; \qquad S = -1 \tag{4}$$

must contain a strange baryon.

When both source densities correspond to strong interactions, i.e. if they are of the well known form [8]

$$S(x) \sim \overline{\psi}_N(x)\,\gamma_5\,\psi(x)\,, \tag{5a}$$

then all states are nucleon states (i.e. nuclei) with no strangeness. In that case, one can replace $|\,\Lambda\,\rangle$ with a nucleon state $|\,i\,\rangle$. The general expression (1) can be easily specialized for the strong interaction case (see Eq. (16) below). In the expression (1), one can first carry out the integration over times (x_0, y_0) using the relation (2). A useful identity is [8,10]

$$\langle\, f\,|\,S(x)\,|\,n\,\rangle = \mathrm{e}^{\mathrm{i}(E_f - E_n)x_0}\langle\, f\,|\,S(\vec{x})\,|\,n\,\rangle. \tag{5b}$$

An analogous expression holds for other matrix elements. Also

$$\begin{aligned} \mathrm{d}x^0\,\mathrm{d}y^0 &= 2\,\mathrm{d}\xi\,\mathrm{d}\eta, \\ x^0 &= \xi + \eta, \\ y^0 &= -\xi + \eta. \end{aligned} \tag{6}$$

The integration $\mathrm{d}\eta$ can be immediately carried out which results in

$$A_{f\Lambda} = \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}\xi \int \mathrm{d}^4k \frac{\mathrm{e}^{-2\mathrm{i}\xi k_0 + \mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2 - \mu^2 + \mathrm{i}\,\epsilon} 2\pi\delta(E_f - E_\Lambda)$$

$$\times \left[\theta(\xi)\sum_n \mathrm{e}^{\mathrm{i}\Delta_n\xi}\alpha_n(\vec{x},\vec{y}) + \theta(-\xi)\sum_s \mathrm{e}^{-\mathrm{i}\Delta_s\xi}\beta_s(\vec{y},\vec{x})\right]. \tag{7a}$$

Here

$$\Delta_i = E_f + E_\Lambda - 2E_i,$$
$$\alpha_n = \langle\, f \,|\, S(\vec{x}) \,|\, n \,\rangle\langle\, n \,|\, W(\vec{y}) \,|\, \Lambda \,\rangle, \tag{7b}$$
$$\beta_s = \langle\, f \,|\, W(\vec{y}) \,|\, s \,\rangle\langle\, s \,|\, S(\vec{x}) \,|\, \Lambda \,\rangle,$$

and E_k are relativistic energies.

Integrating over $\mathrm{d}\xi$, one obtains

$$\begin{aligned} A_{f\Lambda} &= 2\pi \mathrm{i}\, \delta(E_f - E_\Lambda) \int \mathrm{d}^3x\, \mathrm{d}^3y \int \mathrm{d}^4k \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2 - \mu^2 + \mathrm{i}\,\epsilon} \\ &\times \left[\sum_n \alpha_n \frac{1}{\Delta_n - 2k_0 + \mathrm{i}\,\epsilon} + \sum_s \beta_s \frac{1}{\Delta_s + 2k_0 + \mathrm{i}\,\epsilon} \right]. \end{aligned} \tag{8}$$

Integration in the complex k_0 plane over the contours shown in Fig. 1 leads to

$$\begin{aligned} A_{f\Lambda} &= 2\pi^2 \delta(E_f - E_\Lambda) \int \mathrm{d}^3x\, \mathrm{d}^3y \int \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \\ &\times \left[\sum_n \frac{\alpha_n(\vec{x},\vec{y})}{\omega(\Delta_n - 2\omega)} + \sum_s \frac{\beta_s(\vec{y},\vec{x})}{\omega(\Delta_s - 2\omega)} \right], \\ \omega &= \sqrt{\vec{k}^{\,2} + \mu^2}\,. \end{aligned} \tag{9}$$

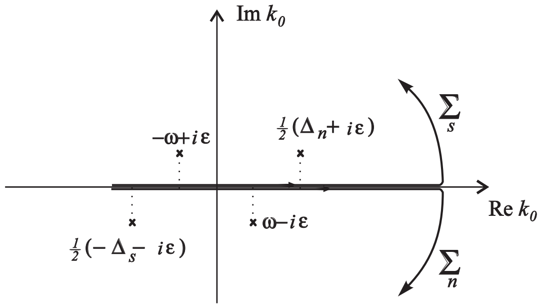

Fig. 1. The contours in the k_0 plane. Here $\omega^2 = \vec{k}^2 + \mu^2$.

In the nonrelativistic limit, the energy differences Δ_i can be approximated by the corresponding baryon mass differences. Schematically one can use the following baryonic contents:

$\|\, f \,\rangle$	K nucleons,
$\|\, \Lambda \,\rangle$	$1\Lambda + (K-1)$ nucleons,
$\|\, s \,\rangle$	$1\Lambda + (K-1)$ nucleons,
$\|\, n \,\rangle$	K nucleons.

Thus

$$
\begin{aligned}
E_f &\to K \cdot m_N, \\
E_\Lambda &\to (K-1) \cdot m_N + m_\Lambda, \\
E_s &\to (K-1) \cdot m_N + m_\Lambda, \\
E_n &\to K \cdot m_N, \\
\Delta_s &= E_f + E_\Lambda - 2E_s \to -2\delta, \\
\Delta_n &= E_f + E_\Lambda - 2E_n \to 2\delta, \\
\delta &= (m_\Lambda - m_N)/2.
\end{aligned} \tag{10}
$$

In such an approximation, the factors depending on ω and Δ_i can be taken out of the summations in Eq. (9). In the nonrelativistic treatment [9], one finds $\sum_n \alpha_n = \sum_n \beta_n = \langle f \,|\, O(x,y) \,|\, \Lambda \rangle$. Thus one obtains

$$
\begin{aligned}
A_{f\,\Lambda} &= 2\pi^2\, \delta(E_f - E_\Lambda) \int \mathrm{d}^3x\, \mathrm{d}^3y \int \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \\
&\quad \times \frac{1}{2}\left[-\frac{1}{\omega(\delta+\omega)} + \frac{1}{\omega(\delta-\omega)} \right] \langle f \,|\, O(\vec{x},\vec{y}) \,|\, \Lambda \rangle, \\
O(\vec{x},\vec{y}) &= S(\vec{x})\, W(\vec{y}).
\end{aligned} \tag{11}
$$

The integration over $\vec{k}$ gives the *shifted* Yukawa function [5]

$$
\int \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{2}\left[-\frac{1}{\omega(\delta+\omega)} + \frac{1}{\omega(\delta-\omega)} \right] = -\int \mathrm{d}^3k\, \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{\omega^2-\delta^2} = -2\pi^2 \frac{\mathrm{e}^{-\kappa r}}{r},
$$

$$
r = |\vec{x}-\vec{y}|, \qquad \kappa = \sqrt{\mu^2-\delta^2} = \sqrt{\mu^2 - \frac{1}{4}(m_\Lambda - m_N)^2}. \tag{12}
$$

Eventually, one finds a generic form

$$
A_{f\Lambda} = -4\pi^4 N \delta(E_f - E_\Lambda) \int \mathrm{d}^3x\, \mathrm{d}^3y \frac{\mathrm{e}^{-\kappa r}}{r} \langle f \,|\, O(\vec{x},\vec{y}) \,|\, \Lambda \rangle. \tag{13}
$$

When one deals with the strong interactions among the nucleons, then all states (10) contain nucleons only. One has

$$
\begin{gathered}
E_f,\ E_\Lambda \equiv E_i, \qquad E_s,\ E_n \to K m_N, \\
\Delta_s = \Delta_n = 0, \\
\delta = 0,
\end{gathered} \tag{14a}
$$

consequently

$$
\kappa \to \mu \tag{14b}
$$

and one finds the well known form [8]

$$
V_Y(r) = \frac{\mathrm{e}^{-\mu r}}{r}. \tag{15}
$$

More precisely, one should start from the formula (9) and write it in the limit (10b) and (14a) as

$$A_{fi} = 2\pi^2\,\delta(E_f - E_i)\int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}^3k\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$\times\left[\sum_n \hat{\alpha}_n\left(-\frac{1}{2\omega^2}\right) + \sum_s \hat{\beta}_s\left(-\frac{1}{2\omega^2}\right)\right] \tag{16a}$$

with

$$\hat{\alpha} = \langle\, f \,|\, S(\vec{x}) \,|\, n \,\rangle\langle\, n \,|\, S(\vec{y}) \,|\, i \,\rangle \qquad \text{and}$$

$$\hat{\beta} = \langle\, f \,|\, S(\vec{y}) \,|\, s \,\rangle\langle\, s \,|\, S(\vec{x}) \,|\, i \,\rangle. \tag{16b}$$

Integration over d^3k gives

$$A_{fi} = -2\pi^4\,\delta(E_f - E_i)\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{\mathrm{e}^{-\mu r}}{r}\left[\sum_n \hat{\alpha}_n + \sum_s \hat{\beta}_s\right]. \tag{16c}$$

If one was interested in the strong pion exchange [8] only, one would start directly with formula (7) neglecting k_0 in the first denominator and using Eqs. (14a) and (16b). Only the first factor in thus redefined (7) depends on k_0. One finds

$$\int \mathrm{d}k_0\,\mathrm{e}^{-2\mathrm{i}\xi k_0}\int \mathrm{d}^3k\,\frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{-\,(\vec{k}^2+\mu^2)+\mathrm{i}\,\epsilon} \quad = \quad -2\pi^3\delta(\xi)\frac{\mathrm{e}^{-\mu r}}{r} \tag{17}$$

and ends with

$$A_{fi} \quad \rightarrow \quad -4\pi^4\delta(E_f - E_i)\int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}\xi\,\frac{\mathrm{e}^{-\mu r}}{r}$$

$$\times\delta(\xi)\left[\sum_n \theta(\xi)\hat{\alpha}_n + \sum_s \theta(-\xi)\hat{\beta}_s\right]. \tag{18}$$

The formula (16c) can be equal to (18) only if[1]

$$\int \delta(\xi)\theta(\xi)\,\mathrm{d}\xi = \int \delta(-\xi)\theta(\xi)\,\mathrm{d}\xi \quad = \quad \frac{1}{2}. \tag{19}$$

This corresponds to the identity which can be read from Ref. [1]. In the equal-mass limit (14), (17), $|\,\Lambda\,\rangle = |\,i\,\rangle$ the previous result (16) can be considered as leading to the identity (19), which is a consequence of the generalized delta function definition [1].

[1] The second term in (19) follows from the first by a trivial coordinate change $\xi \rightarrow -\xi$.

A very interesting result follows if terms which were neglected in (7) in order to obtain Eq. (18) are marked by the parameter λ. For that purpose, one writes Eq. (7a) as

$$A_{fi} = \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}\xi \int \mathrm{d}^3k\,\mathrm{d}k_0 \frac{\mathrm{e}^{-2\mathrm{i}\xi k_0+\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{\lambda^2 k_0^2-\omega^2+\mathrm{i}\,\epsilon} 2\pi\delta(E_f-E_i) \times \left[\theta(\xi)\sum_n \mathrm{e}^{\mathrm{i}\lambda\Delta_n\xi}\hat{\alpha}_n(\vec{x},\vec{y}) + \theta(-\xi)\sum_s \mathrm{e}^{-\mathrm{i}\lambda\Delta_s\xi}\hat{\beta}_s(\vec{x},\vec{y})\right]. \tag{20}$$

The parameter λ determines the sequence which for $\lambda \to 0$ can be understood as and equivalent to the integration over distributions appearing in Eq. (18). Keeping λ finite, the integrations over ξ and k_0 in the formula (20) can be carried out explicitly. Instead of Eq. (9), one finds

$$A_{fi}(\lambda) = 2\pi^2\delta(E_f-E_i)\int \mathrm{d}^3x\,\mathrm{d}^3y \int \mathrm{d}^3k\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \times \left[\sum_n \frac{\hat{\alpha}_n(\vec{x},\vec{y})}{\omega(\lambda^2\Delta_n-2\omega)} + \sum_s \frac{\hat{\beta}_s(\vec{x},\vec{y})}{\omega(\lambda^2\Delta_s-2\omega)}\right]. \tag{21a}$$

This posseses the well defined limit for $\lambda \to 0$ which is shown in Eq. (16), i.e.

$$\lim_{\lambda\to 0} A_{fi}(\lambda) = A_{fi}\ (\text{Eq.}\,(16)). \tag{21b}$$

3. *Summary*

In all above derivations and comparisons, the discussion of either the weak or the strong interactions was immaterial. Only very general physical properties, common to any physical process, the time dependence and the energy conservation, are essential. That led to the physical expressions which modelled the mathematical description of “improper” functions [2,3,11] as distributions [12]. All what was needed was the introduction of a different mass baryon. That can be considered as a purely mathematical device which allows to have a good sequence of functions which in the limit leads to a delta function identity. It is gratifying for a physicist that such a selection corresponds to a well known (i.e. hypernuclei decay) case [9]. Formally that can be expressed through the parameter λ dependent expression (21), i.e. m_Λ in (10b). The limit $\lambda \to 0$ in Eq. (21) exists and that expression can be understood as a defining a sequence of functions which lead to the relation (17). Furthermore, the omission of the terms in Eq. (21) marked by λ can be justified by physical considerations. When one starts directly from the modified formula (7) (i.e., $k^2-\mu^2 \to -(\vec{k}^{\,2}+\mu^2)$), the integration over k_0 simply defines the Dirac delta function (17). No complex plane k_0 integration (Fig. 1) is ever needed.

Presented results illustrate the general Dirac’s statement about the usefulness of delta-functions [2]: *“Therefore it should be possible to rewrite the theory in a* *form in which the improper functions appear all through only in integrands... The use of improper functions thus does not involve any lack of rigour in the theory but is merely a convenient notation, enabling us to express in a concise form certain relations which we could, if necessary, rewrite in a form not involving improper functions, but only in a cumbersome way which tend to obscure the argument."* It is interesting to note how the mathematical contributions of Hevyside and Dirac [2], inspired by physical processes, were perceived by a famous mathematician [13]. He wrote *"... I heard of the Dirac function for the fist time in my second year at ENS. ... which absolutely disgusted us, but it is true that those formulas were so crazy from the mathematical point of view that there was simply no question of accepting them. ... nine years later, I discovered distributions. ... This at least can be deduced from the whole story: it's good thing that theoretical physicists do not wait for mathematical justification before going ahead with their theories."*

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FORMALIZAM IZMJENE MEZONA I DEFINICIJA DELTA FUNKCIJE

Pokazujemo da u okviru elementarnih jedno-mezonskih izmjena, upotreba "nepravih" delta "funkcija" može davati fizički ispravne rezultate. Postigli smo iste rezultate uz primjenu graničnih vrijednosti integrala preko "pravih" funkcija, i tako pripremili primjere nizova povezanih s delta "funkcijama". Ta formulacija nizova integrala sama proizlazi iz fizičkih razmatranja hipernuklearnih procesa.