ISSN 1330–0016
CODEN FIZBE7

# A STUDY OF GLOBAL STRING IN LYRA GEOMETRY

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Received 30 January 2003; revised manuscript received 5 May 2005
Accepted 6 June 2005 Online 6 February 2005

An exact solution for the space-time outside the core of a $U(1)$ static global string in Lyra geometry is presented. Our global string is nonsingular at a finite distance from the string core. The gravitational field of the global string is shown to be attractive in nature.

PACS numbers: 98.80cq, 04.20 jb, 04.50 UDC 52.423
Keywords: Global string, Lyra geometry, gravitational field

## 1. Introduction

The origin of the structure in the Universe is one of the greatest cosmological mysteries even today. It is believed that the early Universe had undergone a number of phase transitions as it cooled down from its hot initial state [1]. Phase transitions in the early Universe could have given rise to various forms of topological defects.

A defect is a discontinuity in the vacuum, and depending on the topology of the vacua, the defects could be domain walls, cosmic strings, monopoles and textures [2]. Among these defects, cosmic strings have received particular attention, mainly because of their cosmological implications. The double quasar problem can be well explained by strings, and galaxy formation might also be generated by density fluctuation in the early Universe due to the strings [3]. Strings are of two general types: local and global. These are said to be local or global depending on their origin from the breakdown of local or global $U(1)$ symmetry.

There has been much discussions [4] on the gravitational field of different types of strings, beginning with the work of Vilenkin [3]. Local strings have vanishing energy and momentum outside the core of the string. The space-time around an infinitely-long local string is Minkowskian space-time minus a wedge. However, global strings have a non-trivial Goldstone boson field extending beyond the core, giving rise to non-zero momentum tensors throughout the space. For such a string, Harrari and Sikivie [4] presented a solution of the linearised field equations, neglecting the radial variation of the scalar field outside the core of the string. But their solution is not self-consistent, as it became evident since there is a physical singularity at a finite distance from the string core.

As far as we know, the solutions for the full non-linear Einstein equations in the case of a global string have been obtained by Cohen and Kaplan [5] and by Sen and Banerjee [6].

In last few decades, there has been considerable interest in alternative theories of gravitation. The most important among them are the scalar-tensor theories proposed by Lyra [7] and by Brans and Dicke [7].

Lyra [7] proposed a modification Riemannian geometry by introducing a gauge function in to the structureless manifold that bears a close resemblance to the Weyls geometry.

In general relativity Einstein, succeeded in geometrising gravitation by identifying the metric tensor with the gravitational potentials.

In the scalar-tensor theory of Brans and Dicke, on the other hand, scalar field remains alien to the geometry. Lyras geometry is more in keeping with the spirit of Einsteins principle of geometrisation, since both the scalar and tensor fields have more or less intrinsic geometrical significance. In the subsequent investigations, Sen [8] and Sen and Dunn [8] proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equation based on Lyras geometry, which in normal gauge may be written as

$$R_{ik} - \frac{1}{2} g_{ik} R + \frac{3}{2} \phi_i \phi_k - \frac{3}{4} g_{ik} \phi_m \phi^m = -8\pi T_{ik} \,, \tag{1}$$

where $\phi_i$ is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [10] pointed out that the constant displacement field $\phi_i$ in Lyras geometry plays the role of the cosmological constant $\Lambda$ in the normal general relativistic treatment. According to Halford, the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations. Soleng [9] has pointed out that the constant displacement field in Lyras geometry will either include a creation field and be equal to Hoyles creation-field cosmology, or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

Subsequent investigations were done by several authors in scalar-tensor theory and cosmology within the framework of Lyra geometry [10].

Recently, Rahaman has obtained weak-field approximate solutions of global string in Lyra geometry [11]. In the present work, we have been able to find an exact solution for the space-time outside of a global string within the framework of Lyra geometry in normal gauge, i.e., displacement vector $\phi_i = (\beta, 0, 0, 0)$ where $\beta$

is a constant. To achieve such a solution, we have relaxed the condition of a Lorenz boost that was admitted along the symmetry axis of the string.

## 2. The basic equations

To describe the space-time geometry due to an infinitely-long static cosmic string, the line element is taken to be the general static cylindrical symmetric one, given by

$$\mathrm{d}s^2 = \mathrm{e}^{2(k-u)}\,(\mathrm{d}t^2 - \mathrm{d}r^2) - \mathrm{e}^{2u}\mathrm{d}z^2 - \mathrm{e}^{-2u}w^2\mathrm{d}\theta^2\,, \tag{2}$$

where $k$, $u$, $w$ are functions of $r$ alone.

For the global string, the energy-momentum tensor components are calculated from the action density of a scalar field $\Psi$ along with a Mexican hat potential

$$L = \frac{1}{2}g^{ab}\Psi^*_{,a}\Psi_{,b} - \frac{\lambda}{4}(\Psi^*\Psi - v^2)^2\,, \tag{3}$$

where $\lambda$ and $v$ are constants and $\delta = (v\sqrt{\lambda})^{-1}$ is a measure of the core radius of the string.

It has been shown that the field configuration can be chosen as

$$\Psi(r) = vf(r)\exp(\mathrm{i}\theta) \tag{4}$$

in the cylindrical field coordinates.

The usual boundary conditions on $f(r)$ are $f(0) = 0$ and $f(r) \to 1$ as $r \to \delta$. As we are interested in space-time outside the core of the string, for our purpose

$$f(r) = 1 \quad \text{and} \quad f^1(r) = 0 \tag{5}$$

is a good approximation.

The non-zero components of the energy momentum tensor outside the core of the string now become

$$T^t_t = T^r_r = T^z_z = -T^\theta_\theta = v^2\frac{\exp(2u)}{2w^2}\,. \tag{6}$$

The gravitational field Eqs. (1) for a global string in Lyra geometry look like

$$-\frac{w^{11}}{w} + \frac{k^1w^1}{w} - u^{12} - \frac{3}{4}\beta^2 = -4\pi v^2\frac{\exp(2k)}{w^2}\,, \tag{7}$$

$$-\frac{k^1w^1}{w} + u^{12} + \frac{3}{4}\beta^2 = -4\pi v^2\frac{\exp(2k)}{w^2}\,, \tag{8}$$

$$-k^{11} - u^{12} + \frac{3}{4}\beta^2 = 4\pi v^2 \frac{\exp(2k)}{w^2}\,, \tag{9}$$

$$-\frac{w^{11}}{w} + 2u^{11} - k^{11} + \frac{2u^1 w^1}{w} - u^{12} + \frac{3}{4}\beta^2 = -4\pi v^2 \frac{\exp(2k)}{w^2} \tag{10}$$

['$^1$' indicates the differentiation w.r.t. '$r$'].

## 3. Solutions

Subtracting Eq. (9) from Eq. (10), we get

$$-\frac{w^{11}}{w} + 2u^{11} = 2\frac{u^1 w^1}{w} = -8\pi v^2 \frac{\exp(2k)}{w^2}\,. \tag{11}$$

Also, (7) and (8) give

$$ww^{11} = 8\pi v^2 \exp(2k)\,. \tag{12}$$

From the above two equations, we get

$$u^{11} + \frac{u^1 w^1}{w} = 0\,. \tag{13}$$

This implies

$$u^1 = s/w \tag{14}$$

(s is an integration constant).

From (14), we get

$$u^{11} = -sw^1/w^2\,. \tag{15}$$

Using (12), (14) and (15), from (7), we get

$$\frac{k^1 w^1}{w} + \frac{s^2}{w^2} + \frac{3}{4}\beta^2 = -4\pi v^2 \frac{\exp(2k)}{w^2}\,. \tag{16}$$

Also, using (12), (14), (15) and (16), we get

$$wk^{11} + k^1 w^1 = (3/2)\beta^2 w = (3/2)\beta^2 \mathrm{d}F/\mathrm{d}r\,, \tag{17}$$

where

$$w = \mathrm{d}F/\mathrm{d}r\,, \tag{18}$$

i.e., $w$ is the primitive function of $F$.

Equation (17) implies

$$k^1 = (3/2)\beta^2 F/w\,. \tag{19}$$

Subtracting (8) from (7) and using (14), (15), (18) and (19), we get the non-linear differential equation for $F$ as

$$2F^1F^{111} - 6\beta^2 FF^{11} + 3\beta^2 F^{12} + 4s^2 = 0\,. \tag{20}$$

We see that the solution of this differential equation is

$$F = \cosh[\sqrt{(3/2)}\,\beta r]\,, \tag{21}$$

provided $s = (3/2)\beta^2$.

Hence, we get the general solutions to the field equations as

$$\begin{aligned} w &= \sqrt{3/2}\;\beta\,\sinh[\sqrt{(3/2)}\,\beta r]\,,\\ u &= \ln\{\tanh[\sqrt{(3/2)}\,\beta r]\}\,,\\ k &= \ln\{\sinh[\sqrt{(3/2)}\,\beta r]\}\,. \end{aligned} \tag{22}$$

In view of the above solutions, the metric can be written as

$$\begin{aligned} \mathrm{d}s^2 &= \sinh^2[\sqrt{(3/2)}\beta r]\coth^2[\sqrt{(3/8)}\beta r](\mathrm{d}t^2 - \mathrm{d}r^2)\\ &- \tanh^2[\sqrt{(3/8)}\beta r]\mathrm{d}z^2 - (3/2)\beta^2\sinh^2[\sqrt{(3/2)}\beta r]\coth^2[\sqrt{(3/8)}\beta r]\mathrm{d}\theta^2\,. \end{aligned} \tag{23}$$

From the metric itself, it is quite apparent that there is no singularity at any finite distance from the core of the string. This is unlike the situation for a global string in general relativity [13].

## 4. Motion of test particle

Suppose a relativistic particle of mass $m$ is moving in the gravitational field described by Eq. (1). The Hamilton - Jacobi (H-J) equation is [6]

$$\exp[2(u-k)]\left[\left(\frac{\delta S}{\delta t}\right)^2 - \left(\frac{\delta S}{\delta r}\right)^2\right] - \exp(-2u)\left(\frac{\delta S}{\delta z}\right)^2 - \frac{\exp(2u)}{w^2}\left(\frac{\delta S}{\delta \theta}\right)^2 + m^2 = 0\,. \tag{24}$$

This equation can be solved easily using the method of separation of variables, and the resulting solution is

$$S(r,z,\theta,t) = -Et + S_1(r) + Mz + J\theta\,, \tag{25}$$

where the constants $E$ and $J$ are identified as the energy and angular momentum of the particle [6]. If we substitute the ansatz (25) for $S$ in the H-J equation (24), then we get

$$S_1(r) = \epsilon \int \left[ E^2 - M^2 \exp(2k - 4u) - J^2 \frac{\exp(2k)}{w^2} + m^2 \exp(2k - 2u) \right]^{1/2} \mathrm{d}r \,. \tag{26}$$

Here $\epsilon = \pm 1$ stands for the change of sign whenever $r$ passes through a zero of the integrand in (26).

We now determine the trajectory of the particle (following the H-J method) by considering [6]

$$\left( \frac{\delta S}{\delta E} \right) = constant, \qquad \left( \frac{\delta S}{\delta J} \right) = constant, \qquad \left( \frac{\delta S}{\delta M} \right) = constant$$

(without any loss generality, one can consider the constants to be zero).

As a result, we find

$$t = \epsilon \int \frac{\exp(2k)}{w^2} \left[ E^2 - M^2 \exp(2k - 4u) - J^2 \frac{\exp(2k)}{w^2} + m^2 \exp(2k - 4u) \right]^{-1/2} \mathrm{d}r \,, \tag{27}$$

$$\theta = \epsilon \int \left[ E^2 - M^2 \exp(2k - 4u) - J^2 \frac{\exp(2k)}{w^2} + m^2 \exp(2k - 4u) \right]^{-1/2} \mathrm{d}r \,, \tag{28}$$

$$z = \epsilon \int \exp(2k - 4u) \left[ E^2 - M^2 \exp(2k - 4u) - J^2 \frac{\exp(2k)}{w^2} + m^2 \exp(2k - 4u) \right]^{-1/2} \mathrm{d}r \,. \tag{29}$$

From (27), the radial velocity of the particle is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{E^2 - M^2 \exp(2k - 4u) - J^2 (\exp(2k)/w^2) + m^2 \exp(2k - 4u)}{\exp(2k)/w^2} \,. \tag{30}$$

The turning point of the trajectory (where $\mathrm{d}r/\mathrm{d}t = 0$) is

$$\frac{E}{m} = \left[ \frac{M^2}{m^2} \exp(2k - 4u) - J^2 \frac{\exp(2k)}{w^2 m^2} - \exp(2k - 4u) \right]^{1/2} , \tag{31}$$

and this determines the potential curve.

The extremes of the potential curve are given by

$$2\cosh\left[\frac{M^2}{m^2}\coth^4\frac{kr}{2}-\frac{J^2}{m^2}-\coth^2\frac{kr}{2}\right] \tag{32}$$

$$+\sinh(kr)\left[-2\frac{M^2}{m^2}\coth^3\frac{kr}{2}+\coth\frac{kr}{2}\right]\mathrm{cosech}^2\frac{kr}{2}=0$$

where $k=\sqrt{(3/2)}\beta$. It is evident that Eq. (32) has real solutions. So orbit of a massive test particle is bounded, i.e., the particle can be trapped by the global string in Lyra geometry. In other words, the global string exerts attractive gravitational effects on the test particles.

## 5. *Concluding remarks*

We have shown that the global string is consistent in Lyra geometry. We have obtained the exact solutions for the space-time metric of the global string. The solution, represented by Eq. (23), exhibits no singularity at a finite distance from the string core. But in general relativity all solutions have singularities at finite values of $r$ [13]. The study of motion of test particles reveals that the global string in Lyra geometry exerts gravitational force which is attractive in nature. It is dissimilar to the case of a global string in general relativity which gives rise to a repulsive effect at a certain distance from the core [4]. So for future work it will be interesting to study different properties of different topological defects based on Lyra geometry.

*Acknowledgements*

I am thankful to IUCAA for sending papers and preprints.

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## PROUČAVANJE GLOBALNIH STRUNA U LYRINOJ GEOMETRIJI

Dajemo egzaktno rješenje za prostor-vrijeme izvan sredice mirne globalne $U(1)$ strune u Lyrinoj geometriji. Nalazi se da globalna struna nije singularna na konačnoj daljini od sredice strune. Pokazuje se da je gravitacijsko polje globalne strune privlačno.