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THE MASSLESS SPIN-1 PARTICLES IN THE ROTATING SPACE-TIMES

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We consider the massless Duffin–Kemmer–Petiau equation for the general rotating space-times, then find its second-order form for a given geometry. Using this second-order differential equation for two well-known cosmological models, the exact solutions of the massless Duffin–Kemmer–Petiau equation were obtained. On the other hand, by using spinor form of the Maxwell equations, the propagation problem is reduced to the solution of the second-order differential equation of a complex combination of the electric and magnetic fields. For these two different approaches, we obtain the spinors in terms of the field-strength tensor.

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1. *Introduction*

To understand our real physical Universe, it is necessary to know solutions of both the Einstein and the quantum field equations to discuss the dynamics of the Universe and its particles. The passage from the field equations of relativistic quantum mechanics in space-time to the general relativistic quantum field equations can be done by using the principle of covariance and the tetrad formalism according to the Tetrode–Weyl–Fock–Ivanenko procedure [$1-9$], expanded to include spin transformation quantities. Since the gravitational effects are weak, it seems that general relativistic wave equations are not important on the atomic scale, but for many astrophysical situations, one has to take into account gravitational effects due to their dominant role. One of the most fascinating aspects of the gravitational effects is their evident role in particle creation. To construct a steady theory of quantum field theory in curved space-time, it is necessary to analyze the single-particle states, since these states are examined to get the dynamics of the particles in a given background. The curved space-time quantum field theory provides a strong motivation for a unified theory of gravitation and quantum mechanics.

Electromagnetic fields are described by the Maxwell equations. Following a minimal coupling procedure, the scalar products are performed with the Riemannian metric $g_{\mu\nu}$, and the partial derivatives are replaced by covariant ones, then Maxwell equations can be written in general relativity [10]. In the literature, however, there exist many attempts to pass down from the classical wave theory of light to quantum mechanics [11]. If complex combinations of the electric and magnetic fields are taken as the elements of a three-component spinor, the Maxwell curl equations can be synthesized into a form similar to that of the Weyl equation for the neutrino. In these three-component formulations, the divergence equations are imposed as constraint equations. Furthermore, these are valid only in free space in the absence of any source. These two deficiencies have been put right in the work of Moses [12], who found a four-component spinor formulation which casts the Maxwell equations in the form of a massless Dirac equation. He united the source in the form of a four-component spinor and combined the four Maxwell equations in the presence of a source.

Much earlier than the above attempts, Duffin, Kemmer and Petiau had formulated the wave equation (the DKP equation) for massive spin-1 particles [13, 14]. They showed that the first-order form of the Klein–Gordon and Proca field equations can be represented in the Dirac-like matrix form

$$\big(\mathrm{i}\beta^{(k)}\partial_{(k)} - m\big)\Psi = 0\,, \tag{1}$$

where β matrices satisfy the following relation

$$\beta^{(a)}\beta^{(b)}\beta^{(c)} + \beta^{(c)}\beta^{(b)}\beta^{(a)} = \beta^{(a)}\delta^{(b)(c)} + \beta^{(c)}\delta^{(b)(a)}. \tag{2}$$

Equation (1) is a first-order equation for spin-0 and spin-1 bosons, in contrast to the other relativistic wave equations for bosons. Lately, the applications of the Duffin–Kemmer–Petiau (DKP) theory to quantum chromo-dynamics (QCD) have been considered by Gribov [15]. Additionally, it has been used to find covariant Hamiltonian dynamics by Kanatchikov [16]. Within the framework of general relativity, the DKP equation has been conformed to curved space-time by Red'kov [17] and Lunardi et al. [18]. With the generalization of DKP equation to the curved space-time, it has become important to investigate the behavior of bosons in curved backgrounds. The covariant form of DKP equation is given by

$$\big(\mathrm{i}\beta^{\mu}\nabla_{\mu} - m\big)\Psi = 0\,, \tag{3}$$

where

$$\beta^{\mu}(x) = \gamma^{\mu}(x) \otimes I + I \otimes \gamma^{\mu}(x) \tag{4}$$

are the Kemmer matrices in curved space-time and they are related to flat Minkowski space-time as $\beta^{\mu}(x) = b^{\mu}_{(i)}\widetilde{\beta}^{(i)}$ with a tetrad frame that satisfies

$g_{\mu\nu} = b^{(i)}_\mu b^{(j)}_\nu \eta_{(i)(j)}$. The covariant derivative in Eq. (3) is $\nabla_\mu = \partial_\mu - \Omega_\mu$ with spinorial connections which can be written as

$$\Omega_\mu = \Gamma_\mu \otimes I + I \otimes \Gamma_\mu \,, \tag{5}$$

where

$$4\Gamma_\lambda = g_{\mu\alpha}\big[\big(\partial_\lambda a^{(k)}_\nu\big) b^\alpha_{(k)} - \Gamma^\alpha_{\nu\lambda}\big] S^{\mu\nu}. \tag{6}$$

The Christoffel symbols and spin tensor can be written as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\big)\,, \qquad S^{\mu\nu} = \big[\gamma^\mu, \gamma^\nu\big]\,, \tag{7}$$

where $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}$ and γ^μ are the Dirac matrices in curved space-time and they are related to at Minkowski space-time as

$$\gamma^\mu(x) = b^\mu_{(i)} \widetilde{\gamma}^{(i)}. \tag{8}$$

The Dirac-like equation (3) can be solved by using standard techniques used for the Dirac equation.

The counterpart of the Maxwell equations in general relativistic quantum mechanics can be obtained as the zero-mass limit of the DKP equation with appropriate identification of the components of the DKP spinor with electromagnetic field strengths. In 1997 Ünal showed that the wave equation of massless spin-1 particle in at space-time is equivalent to free-space Maxwell equations [19]. Then Ünal and Sucu solved the general relativistic massless DKP equations (hereafter referred to as the mDKP equation) in Robertson–Walker space-time written in spherical coordinates [20]. By using the same technique, the mDKP equation had been solved for the stationary Gödel and the Gödel-type space-time and also the non-stationary Gödel-type cosmological universes [21, 22]. In this technique, the Kemmer matrices are written as a direct product of Pauli spin matrices with unit matrix resulting in (4×4) matrices. This representation leads to a spinor which is related to a complex combination of the electric and magnetic fields. Among of the advantages to use the mDKP equation is that its a simple (4×4) matrix form simplifing the solution procedure in comparison with the Maxwell equations. The quantum-mechanical solution is also important in the discussion of the wave-particle duality of electromagnetic fields, since the particle nature of the electromagnetic field can be analyzed only by a quantum-mechanical equation. Furthermore, the mDKP equation removes the unavoidable usage of $(3 + 1)$ space-time splitting formalism for the Maxwell equations mentioned by Saibatalov [23]. The mDKP equation is given as follows

$$\beta^\mu \nabla_\mu \Psi = 0\,, \tag{9}$$

where β^μ are now

$$\beta^\mu(x) = \sigma^\mu(x) \otimes I + I \otimes \sigma^\mu(x)\,, \tag{10}$$

with

$$\sigma^\mu(x) = \big(I, \vec{\sigma}(x)\big)\,. \tag{11}$$

The covariant derivative ∇_μ with spinorial connections Ξ_μ are given with the limit $\gamma^\mu \to \sigma^\mu$ as

$$\nabla_\mu = \partial_\mu - \Xi_\mu = \partial_\mu - \lim_{\gamma\to\sigma} \Gamma_\mu \otimes I + I \otimes \Gamma_\mu\,. \tag{12}$$

In this paper, we investigate the relationship between the classical and quantum theory of light by examining mDKP equation and the Maxwell equations in the general space-times. In that manner we show quantum-mechanical wave function in terms of the Maxwell-field strength-tensor components. Since 4×4 Kemmer matrices have been used, this correspondence can be shown only if a complex combination of the field-strength tensors are used. This paper is organized as follows: in the next section, we define a general metric. In Sec. 3, we give the mDKP equation explicitly and we obtain its second-order form. In Sec. 4 we find the components of the Maxwell field strengths and use them to get second-order differential equation by using spinor formalism. In Sec. V, we obtain exact solution of the mDKP equation in two different cosmological models, and in the last section, we give and discuss some results.

2. *A general rotating cosmological model*

The general spacetime's line element which we choose is given by

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + R^2(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - 2G\mathrm{d}x^0\mathrm{d}x^2, \tag{13}$$

where the functions R and G depend on x^1. This metric describes spatially-homogenous universes with rotation. The line element (13) can be reduced to the known spacetime models under some conditions. We give some space-times which are special cases in Table 1.

TABLE 1. (3+1)-dimensional cosmological models and their line elements.

Space-time	Line-element ($\mathrm{d}s^2$)
Gödel [21]	$-(\mathrm{d}t + \mathrm{e}^{\alpha r}\mathrm{d}\theta)^2 + \mathrm{d}r^2 + \frac{1}{2}(\mathrm{e}^{\alpha r}\mathrm{d}\theta)^2 + \mathrm{d}z^2$
Reboucas [24]	$\mathrm{d}r^2 - (1+3\cosh^2 2r)\mathrm{d}\phi^2 + \mathrm{d}z^2 + 4\cosh 2r\mathrm{d}t\mathrm{d}\phi - \mathrm{d}t^2$
Som–Raychaudhuri [24]	$\mathrm{d}r^2 + r^2(1-r^2)\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2r^2\mathrm{d}t\mathrm{d}\phi - \mathrm{d}t^2$
Minkowski [24]	$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2$
Kantowski–Sachs [25]	$\mathrm{d}x^2 + \sin 2x\mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2$
Soleng [26]	$\mathrm{d}r^2 - 2a\mathrm{d}t\mathrm{d}\phi + (B^2(r+r_0)^2 - a^2)\mathrm{d}\phi^2 + \mathrm{d}z^2 - \mathrm{d}t^2$
Cyl. S. Minkowski [27]	$\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2 - \mathrm{d}t^2$

The matrices of the $g_{\mu\nu}$ and $g^{\mu\nu}$ are defined by

$$\begin{pmatrix} -1 & 0 & -G & 0 \\ 0 & 1 & 0 & 0 \\ -G & 0 & R^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -\frac{R^2}{\Lambda^2} & 0 & -\frac{G}{\Lambda^2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{G}{\Lambda^2} & 0 & \frac{1}{\Lambda^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where $\Lambda^2 = G^2 + R^2$. For the line element given in Eq. (13), the suitably selected tetrads are

$$a_\mu^{(0)} = \delta_\mu^0\,, \qquad a_\mu^{(1)} = \delta_\mu^1\,, \quad a_\mu^{(2)} = G\delta_\mu^0 + \Lambda\delta_\mu^2\,, \qquad a_\mu^{(3)} = \delta_\mu^3\,, \tag{15}$$

$$b_{(0)}^\mu = \delta_0^\mu\,, \qquad b_{(1)}^\mu = \delta_1^\mu\,, \quad b_{(2)}^\mu = \frac{1}{\Lambda}(\delta_2^\mu - G\delta_0^\mu)\,, \quad b_{(3)}^\mu = \delta_3^\mu\,, \tag{16}$$

where $a_{(i)}^\mu = g^{\mu\nu}\eta_{(i)(j)}b_\nu^{(j)}$. The curved Dirac matrices, which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, are given by

$$\gamma^0 = \widetilde{\gamma}^0 - \frac{G}{\Lambda}\widetilde{\gamma}^2, \qquad \gamma^1 = \widetilde{\gamma}^1, \qquad \gamma^2 = \frac{1}{\Lambda}\widetilde{\gamma}^2, \qquad \gamma^3 = \widetilde{\gamma}^3. \tag{17}$$

The non-vanishing Christoffel symbols are

$$\begin{aligned} \Lambda^2\Gamma_{\mu\nu}^0 &= R^2\big[\delta_{\mu\nu}^{11} + R^2\delta_{\mu\nu}^{22} + \delta_{\mu\nu}^{33} - G(\delta_{\mu\nu}^{02} + \delta_{\mu\nu}^{20})\big] + \frac{GG'}{2}(\delta_{\mu\nu}^{01} + \delta_{\mu\nu}^{10}) \\ &\quad + \frac{R}{2}\big(RG' - 2GR'\big)(\delta_{\mu\nu}^{12} + \delta_{\mu\nu}^{21})\,, \end{aligned} \tag{18}$$

$$\begin{aligned} \Lambda^2\Gamma_{\mu\nu}^2 &= R^2\big(\delta_{\mu\nu}^{02} + \delta_{\mu\nu}^{20}\big) + \frac{1}{2}(GG' + 2RR')(\delta_{\mu\nu}^{12} + \delta_{\mu\nu}^{21}) \\ &\quad - \frac{G'}{2}(\delta_{\mu\nu}^{01} + \delta_{\mu\nu}^{10})\,, \end{aligned} \tag{19}$$

$$\Gamma_{\mu\nu}^1 = \frac{G'}{2}(\delta_{\mu\nu}^{02} + \delta_{\mu\nu}^{20}) - RR'\delta_{\mu\nu}^{22}\,, \tag{20}$$

where prime indicates derivative with respect to x^1 and $\delta_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha\delta_\nu^\beta$. The spinorial connections are

$$\Gamma_0 = -\frac{G'}{4\Lambda}\widetilde{\gamma}^{12}, \qquad \Gamma_1 = \frac{G'}{4\Lambda}\widetilde{\gamma}^{20}, \qquad \Gamma_2 = -\frac{G'}{4}\widetilde{\gamma}^{10} + \frac{GG' + 2RR'}{4\Lambda}\widetilde{\gamma}^{12} \tag{21}$$

where we have defined that $\widetilde{\gamma}^{ij} = \widetilde{\gamma}^i\widetilde{\gamma}^j$.

3. The massless spin-1 wave equation

Using standard representation of the Dirac matrices, we obtain the mDKP equation as

$$\begin{aligned}(\widetilde{\beta}^0 - \frac{G}{\Lambda}\widetilde{\beta}^2)\partial_0\Psi + \widetilde{\beta}^1\partial_1\Psi + \frac{1}{\Lambda}\widetilde{\beta}^2\partial_2\Psi + \widetilde{\beta}^3\partial_3\Psi + \frac{\mathrm{i}G'}{4\Lambda}\widetilde{\beta}^{03}\Psi \\ -\frac{\mathrm{i}\Lambda'}{2\Lambda}\widetilde{\beta}^{23}\Psi - \frac{G'}{4\Lambda}(\widetilde{\beta}^{12} - \widetilde{\beta}^{21})\Psi = 0\,,\end{aligned} \tag{22}$$

where $\widetilde{\beta}^{ij} = \widetilde{\beta}^i\widetilde{\beta}^j$. If the following spinor definition is used

$$\Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \tag{23}$$

Eq. (22) gives four coupled first-order differential equations in terms of the components of the spinor as

$$2(\partial_0 + \partial_3)\Psi_0 + \left(\frac{\mathrm{i}G}{\Lambda}\partial_0 + \partial_1 - \frac{\mathrm{i}}{\Lambda}\partial_2\right)(\Psi_1 + \Psi_2) \;=\; 0\,, \tag{24}$$

$$\left(\frac{-\mathrm{i}G}{\Lambda}\partial_0 + \frac{\mathrm{i}}{\Lambda}\partial_2\right)(\Psi_0 - \Psi_3) + 2\partial_0\Psi_1 + \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)(\Psi_0 + \Psi_3) \;=\; 0\,, \tag{25}$$

$$\left(\frac{-\mathrm{i}G}{\Lambda}\partial_0 + \frac{\mathrm{i}}{\Lambda}\partial_2\right)(\Psi_0 - \Psi_3) + 2\partial_0\Psi_2 + \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)(\Psi_0 + \Psi_3) \;=\; 0\,, \tag{26}$$

$$\left(\frac{-\mathrm{i}G}{\Lambda}\partial_0 + \partial_1 + \frac{\mathrm{i}}{\Lambda}\partial_2\right)(\Psi_1 + \Psi_2) + (2\partial_0 - 2\partial_3)\Psi_3 \;=\; 0\,. \tag{27}$$

From Eqs. (25) and (26), it is seen that $\Psi_1 = \Psi_2$. If we use this result, then these four coupled equations reduce to the following three coupled equations

$$2(\partial_0 + \partial_3)\Psi_0 + \left(\frac{\mathrm{i}G}{\Lambda}\partial_0 + \partial_1 - \frac{\mathrm{i}}{\Lambda}\partial_2\right)\Psi_1 \;=\; 0\,, \tag{28}$$

$$\left(\frac{-\mathrm{i}G}{\Lambda}\partial_0 + \frac{\mathrm{i}}{\Lambda}\partial_2\right)(\Psi_0 - \Psi_3) + 2\partial_0\Psi_1 + \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)(\Psi_0 + \Psi_3) \;=\; 0\,, \tag{29}$$

$$\left(\frac{-\mathrm{i}G}{\Lambda}\partial_0 + \partial_1 + \frac{\mathrm{i}}{\Lambda}\partial_2\right)\Psi_1 + (\partial_0 - \partial_3)\Psi_3 \;=\; 0\,. \tag{30}$$

For these three coupled equations, we can choose the spinor as

$$\Psi(x^1) = \exp\{\mathrm{i}(k_2x^2 + k_3x^3 - k_0x^0)\}\varphi(x^1)\,. \tag{31}$$

Thus we have

$$-\mathrm{i}\big(k_0 - k_3\big)\varphi_0 + \left(\frac{k_2 + k_0 G}{\Lambda} + \partial_1\right)\varphi_1 \;=\; 0\,, \tag{32}$$

$$\left(\partial_1 - \frac{k_2 + k_0 G}{\Lambda} + \frac{\Lambda'}{\Lambda}\right)\varphi_0 - 2\mathrm{i}k_0\varphi_1 + \left(\partial_1 + \frac{k_2 + k_0 G}{\Lambda} + \frac{\Lambda'}{\Lambda}\right)\varphi_3 \;=\; 0\,, \tag{33}$$

$$\left(\partial_1 - \frac{k_2 + k_0 G}{\Lambda}\right)\varphi_1 - \mathrm{i}\big(k_0 + k_3\big)\varphi_3 \;=\; 0\,. \tag{34}$$

The components φ_0 and φ_3 can be expressed in terms of φ_1 by

$$\varphi_0 \;=\; \frac{\mathrm{i}}{\mathrm{k}_3 - \mathrm{k}_0}\left(\partial_1 + \frac{k_2 + k_0 G}{M}\right)\varphi_1\,, \tag{35}$$

$$\varphi_3 \;=\; \frac{\mathrm{i}}{\mathrm{k}_3 + \mathrm{k}_0}\left(\frac{k_2 + k_0 G}{M} - \partial_1\right)\varphi_1\,. \tag{36}$$

From the set of above equations, it is found that the second-order differential equation for φ_1 is

$$\left\{\partial_1^2 + \frac{\Lambda'}{\Lambda}\partial_1 + \frac{k_3 G'}{\Lambda} - \left(\frac{k_2 + k_0 G}{\Lambda}\right)^2 + k_0^2 - k_3^2\right\}\varphi_1(x^1) = 0\,. \tag{37}$$

It is easy to find the other components of the spinor from the differential relations given above.

4. *The Maxwell equations*

The propagation of electromagnetic fields has been studied for several reasons. There are many astrophysical situations (light deflection in gravitational lensing, pulsars, quasars, black holes) that involve strong electromagnetic and gravitational fields in interaction. The interaction of electromagnetic and gravitational fields is described by the Maxwell equations in a given curved background. In the absence of electromagnetic source these equations are

$$\frac{1}{\sqrt{-g}}\big(\sqrt{-g}F^{\mu\nu}\big)_{,\nu} \;=\; 0\,, \tag{38}$$

$$F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} \;=\; 0\,, \tag{39}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $A^\mu = (A^0, \vec{A})$, $\vec{A} = \vec{A}(x)$. Here we solve the Maxwell equations for the line element given in (22) to show the correspondence between the mDKP equation and the Maxwell equations. For the particular choice of the functions R and G, the solution of the Maxwell equations has been well done by Saibatalov [23] using similar technique. The contra-variant and covariant field strengths $F^{\mu\nu}$ and $F_{\mu\nu}$ in the general coordinates are

$$\begin{aligned} F^{01} &= E^{(1)} + G(\Lambda)^{-1}B^{(3)}, & F_{01} &= -E^{(1)}, \\ F^{02} &= (\Lambda)^{-1}E^{(2)}, & F_{02} &= -\Lambda E^{(2)}, \\ F^{03} &= E^{(3)} - G(\Lambda)^{-1}B^{(1)}, & F_{03} &= E^{(3)}, \\ F^{12} &= \Lambda^{-1}B^{(3)}, & F_{12} &= GE^{(1)} + \Lambda B^{(3)}, \\ F^{13} &= -B^{(2)}, & F_{13} &= -B^{(2)}, \\ F^{23} &= \Lambda^{-1}B^{(1)}, & F_{23} &= -GE^{(3)} + \Lambda B^{(1)}. \end{aligned} \tag{40}$$

Here $E^{(i)}$ and $B^{(i)}$ are the components of the electric and magnetic fields in the local Lorentz frame.

From Eq. (38), we find the following coupled equations

$$(\Lambda\partial_1 + \Lambda')E^{(1)} + \partial_2 E^{(2)} + \Lambda\partial_3 E^{(3)} - G\partial_3 B^{(1)} + (G\partial_1 + G')B^{(3)} = 0\,, \tag{41}$$

$$\Lambda\partial_0 E^{(1)} + \Lambda\partial_3 B^{(2)} + (G\partial_0 - \partial_2)B^{(3)} = 0\,, \tag{42}$$

$$\partial_0 E^{(2)} - \partial_3 B^{(1)} + \partial_1 B^{(3)} = 0\,, \tag{43}$$

$$-\Lambda\partial_0 E^{(3)} + (G\partial_0 - \partial_2)B^{(1)} + (\Lambda\partial_1 + \Lambda')B^{(2)} = 0\,, \tag{44}$$

and using Eq. (39) we have

$$(G\partial_0 - \partial_2)E^{(1)} + (\Lambda\partial_1 + \Lambda')E^{(2)} + (\Lambda\partial_0)B^{(3)} = 0\,, \tag{45}$$

$$\partial_3 E^{(1)} - \partial_1 E^{(3)} + \partial_0 B^{(2)} = 0\,, \tag{46}$$

$$G\partial_3 E^{(1)} - (G\partial_1 + G')E^{(3)} + (\Lambda\partial_1 + \Lambda')B^{(1)} + \partial_2 B^{(2)} + \Lambda\partial_3 B^{(3)} = 0\,, \tag{47}$$

$$-\Lambda\partial_3 E^{(2)} - (G\partial_0 - \partial_2)E^{(3)} + \Lambda\partial_0 B^{(1)} = 0\,. \tag{48}$$

In terms of the components, these can be written as

$$\begin{aligned} \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)\begin{pmatrix} E^{(1)} \\ B^{(1)} \end{pmatrix} + \frac{1}{\Lambda}\partial_2\begin{pmatrix} E^{(2)} \\ B^{(2)} \end{pmatrix} + \frac{G}{\Lambda}\partial_3\begin{pmatrix} -B^{(1)} \\ E^{(1)} \end{pmatrix} & \\ -\left(\frac{G}{\Lambda}\partial_1 + \frac{G'}{\Lambda}\right)\begin{pmatrix} -B^{(3)} \\ E^{(3)} \end{pmatrix} + \partial_3\begin{pmatrix} E^{(3)} \\ B^{(3)} \end{pmatrix} &= 0\,, \end{aligned} \tag{49}$$

$$\partial_0\begin{pmatrix} -B^{(1)} \\ E^{(1)} \end{pmatrix} + \left(\frac{G}{\Lambda}\partial_0 - \frac{1}{\Lambda}\partial_2\right)\begin{pmatrix} E^{(3)} \\ B^{(3)} \end{pmatrix} + \partial_3\begin{pmatrix} E^{(2)} \\ B^{(2)} \end{pmatrix} = 0\,, \tag{50}$$

$$-\partial_0\begin{pmatrix} -B^{(2)} \\ E^{(2)} \end{pmatrix} + \partial_3\begin{pmatrix} E^{(1)} \\ B^{(1)} \end{pmatrix} - \partial_1\begin{pmatrix} E^{(3)} \\ B^{(3)} \end{pmatrix} = 0\,, \tag{51}$$

$$-\partial_0 \begin{pmatrix} -B^{(3)} \\ E^{(3)} \end{pmatrix} + \left(\frac{G}{\Lambda}\partial_0 - \frac{1}{\Lambda}\partial_2\right) \begin{pmatrix} E^{(1)} \\ B^{(1)} \end{pmatrix}$$
$$+\left(\partial_1 + \frac{\Lambda'}{\Lambda}\right) \begin{pmatrix} E^{(2)} \\ B^{(2)} \end{pmatrix} = 0\,. \tag{52}$$

If we define the complex spinor as

$$H = \begin{pmatrix} H^1 \\ H^2 \\ H^3 \end{pmatrix} = \begin{pmatrix} E^{(1)} + \mathrm{i}B^{(1)} \\ E^{(2)} + \mathrm{i}B^{(2)} \\ E^{(3)} + \mathrm{i}B^{(3)} \end{pmatrix}, \tag{53}$$

the spinor form of the Maxwell equations are found as

$$\left(\partial_1 + \frac{\Lambda'}{\Lambda} + \frac{\mathrm{i}G}{\Lambda}\partial_3\right)H^1 + \frac{1}{\Lambda}\partial_2 H^2 - \mathrm{i}\left(\frac{G}{\Lambda}\partial_1 + \frac{G'}{\Lambda} + \mathrm{i}\partial_3\right)H^3 = 0\,, \tag{54}$$

$$\mathrm{i}\partial_0 H^1 + \partial_3 H^2 + \frac{1}{\Lambda}(G\partial_0 - \partial_2)H^3 = 0\,, \tag{55}$$

$$\partial_3 H^1 - \mathrm{i}\partial_0 H^2 - \partial_1 H^3 = 0\,, \tag{56}$$

$$\frac{1}{\Lambda}(G\partial_0 - \partial_2)H^1 + \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)H^2 - \mathrm{i}\partial_0 H^3 = 0\,. \tag{57}$$

If the following form of the spinor is used

$$H^i(x^0, x^i) = \exp\{\mathrm{i}(k_2x^2 + k_3x^3 - k_0x^0)\}\Pi^i(x^1)\,, \qquad (i = 1, 2, 3)\,, \tag{58}$$

then we find

$$\left(\partial_1 + \frac{\Lambda'}{\Lambda} + \frac{k_3G}{\Lambda}\right)\Pi^1 + \frac{\mathrm{i}k_2}{\Lambda}\Pi^2 - \mathrm{i}\left(\frac{G}{\Lambda}\partial_1 + \frac{G'}{\Lambda} - k_3\right)\Pi^3 = 0\,, \tag{59}$$

$$k_0\Pi^1 + \mathrm{i}k_3\Pi^2 - \mathrm{i}\frac{k_2 + k_0G}{\Lambda}\Pi^3 = 0\,, \tag{60}$$

$$\mathrm{i}k_3\Pi^1 - k_0\Pi^2 - \partial_1\Pi^3 = 0\,, \tag{61}$$

$$-\frac{k_2 + k_0G}{\Lambda}\Pi^1 + \left(\partial_1 + \frac{\Lambda'}{\Lambda}\right)\Pi^2 - k_0\Pi^3 = 0\,. \tag{62}$$

The components Π^1 and Π^2 can be expressed in terms of N^3

$$\Pi^1 = \frac{\mathrm{i}}{\mathrm{k}_0^2 - \mathrm{k}_3^2}\left[k_3\partial_1 + \frac{k_0(k_2 + k_0G)}{\Lambda}\right]\Pi^3, \tag{63}$$

$$\Pi^2 = \frac{1}{k_3^2 - k_0^2}\left[k_0\partial_1 + \frac{k_3(k_2 + k_0G)}{\Lambda}\right]\Pi^3\,. \tag{64}$$

From the set of above equations, it is found that the second-order differential equation for Π^3 is

$$\left\{\partial_1^2 + \frac{\Lambda'}{\Lambda}\partial_1 + \frac{k_3 G'}{\Lambda} - \left(\frac{k_2 + k_0 G}{\Lambda}\right)^2 + k_0^2 - k_3^2\right\}\Pi^3(x^1) = 0\,. \tag{65}$$

This is the same as Eq. (37), and their solution is exactly the same. By comparing Eqs. (35)–(36) and (63)–(64), one can obtain the relations between spinor components of the mDKP equation and the Maxwell equations as follows

$$\varphi_0 = -\Pi^1 + \mathrm{i}\Pi^2, \qquad \varphi_1 = \varphi_2 = \Pi^3, \quad \varphi_3 = \Pi^1 + \mathrm{i}\Pi^2. \tag{66}$$

In terms of the electric and magnetic fields, which are given as

$$E^{(i)} = b_\mu^{(0)} b_\nu^{(i)} F^{\mu\nu}, \qquad B^{(i)} = b_\mu^{(i)} b_\nu^{(0)} \widetilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} b_\mu^{(i)} b_\nu^{(0)} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}\,, \tag{67}$$

the components of the spinor of the mDKP equation are found that

$$\begin{aligned} \varphi_0 &= \mathrm{i}\Lambda(F^{02} + \mathrm{i}\widetilde{F}^{20}) - F^{01} - \mathrm{i}\widetilde{F}^{10} - G(F^{21} + \mathrm{i}\widetilde{F}^{12})\,, \\ \varphi_1 = \varphi_2 &= F^{03} + \mathrm{i}\widetilde{F}^{30} + G(F^{23} + \mathrm{i}\widetilde{F}^{32})\,, \\ \varphi_3 &= \mathrm{i}\Lambda(F^{02} + \mathrm{i}\widetilde{F}^{20}) + F^{01} + \mathrm{i}\widetilde{F}^{10} + G(F^{21} + \mathrm{i}\widetilde{F}^{12})\,. \end{aligned} \tag{68}$$

5. *Exact solution of the mDKP equation*

5.1. *Solution in the Soleng space-time*

Using Eq. (37), second-order form of the mDKP equation in this space-time becomes

$$\left\{\partial_r^2 + \frac{1}{r + r_0}\partial_r - \left(\frac{k_2 + a k_0}{B(r + r_0)}\right)^2 + k_0^2 - k_3^2\right\}\varphi_1(r) = 0\,, \tag{69}$$

where a, r_0 and B are constants. For this second-order differential equation, if we use $(r + r_0)\alpha^{1/2} = \nu$ coordinate transformation (where $\alpha = k_0^2 - k_3^2$) and define $[(k_2 + ak_0)/B]^2 = \beta^2$, than the mDKP equation takes the form

$$\left\{\partial_r^2 + \frac{1}{\nu}\partial_r + 1 - \frac{\beta^2}{\nu^2}\right\}\varphi_1(r) = 0\,. \tag{70}$$

This is the well-known Bessel equation, and from this point of view the exact solution is becomes

$$\Psi_1 = \exp\{\mathrm{i}(k_2\phi + k_3 z - k_0 t)\}\left[A j_\beta(\nu) + B N_\beta(\nu)\right]. \tag{71}$$

From Eqs. (68), the components of the spinor of the mDKP equation in the Soleng space-time are found as

$$\begin{aligned} \varphi_0 &= \mathrm{i}B(r+r_0)(F^{02}+\mathrm{i}\widetilde{F}^{20}) - F^{01} - \mathrm{i}\widetilde{F}^{10} - a(F^{21}+\mathrm{i}\widetilde{F}^{12})\,, \\ \varphi_1 = \varphi_2 &= F^{03}+\mathrm{i}\widetilde{F}^{30} + a(F^{23}+\mathrm{i}\widetilde{F}^{32})\,, \\ \varphi_3 &= \mathrm{i}B(r+r_0)(F^{02}+\mathrm{i}\widetilde{F}^{20}) + F^{01} + \mathrm{i}\widetilde{F}^{10} + a(F^{21}+\mathrm{i}\widetilde{F}^{12})\,. \end{aligned} \tag{72}$$

5.2. *Solution in the cylindrically symmetric Minkowski space-time*

Considering Eq. (37), the second-order form of the mDKP equation in this space-time takes the following form

$$\left\{\partial_r^2 + \frac{1}{r}\partial_r - \frac{k_2^2}{r^2} + k_0^2 - k_3^2\right\}\varphi_1(r) = 0\,. \tag{73}$$

If we use the coordinate transformation $\nu = \alpha^{1/2} r$ with $\alpha = k_0^2 - k_3^2$, than we obtain following well-known equation

$$\left\{\partial_r^2 + \frac{1}{\nu}\partial_r + 1 - \frac{k_2^2}{\nu^2}\right\}\varphi_1(r) = 0. \tag{74}$$

This is the Bessel equation, and the solutions are

$$\Psi_1 = \exp\{\mathrm{i}(k_2\phi + k_3 z - k_0 t)\}\left[A j_{\mp k_2}(\nu) + B N_{\mp k_2}(\nu)\right]. \tag{75}$$

Using Eq. (68), the corresponding components of the spinor of the mDKP equation in the cylindrically symmetric Minkowski space-time are

$$\begin{aligned} \varphi_0 &= \mathrm{i}r(F^{02}+\mathrm{i}\widetilde{F}^{20}) - F^{01} - \mathrm{i}\widetilde{F}^{10}, \\ \varphi_1 = \varphi_2 &= F^{03}+\mathrm{i}\widetilde{F}^{30}, \\ \varphi_3 &= \mathrm{i}r(F^{02}+\mathrm{i}\widetilde{F}^{20}) + F^{01} + \mathrm{i}\widetilde{F}^{10}. \end{aligned} \tag{76}$$

6. *Oscillating regions of the massless spin-1 particles*

Since it is not aimed here to solve Eq. (37) exactly for the given spacetimes in introduction, we will restrict ourselves to discuss how one can obtain the frequency spectrum of the photon by using some models of our general spacetimes. A general method to find the frequency spectrum is to impose the condition on functions which are the solutions of the differential equation. The functions obtained must be bounded for all values as is usually done in quantum mechanics. This procedure gives the quantization of frequency. If the function G vanishes, the line element

(22) reduces an expanding model, and one might expect to obtain the gravitational red shift in frequency. But G is not zero, and this model represents both expansion and rotation.

We introduce a new function of the form

$$\varphi_1(x^1) = \Lambda^{1/2}(x^1)\chi_1(x^1)\,. \tag{77}$$

Using this denition in Eq. (37), we get the following form

$$\left\{\partial_1^2 + \frac{\Lambda''}{2\Lambda} + \left(\frac{\Lambda'}{2\Lambda}\right)^2 + \frac{k_3 G'}{\Lambda} - \left(\frac{k_2 + k_0 G}{\Lambda}\right)^2 + k_0^2 - k_3^2\right\}\chi_1(x^1) = 0\,. \tag{78}$$

From this equation we can write

$$\omega^2(\xi) = \frac{\Lambda''}{2\Lambda} + \left(\frac{\Lambda'}{2\Lambda}\right)^2 + \frac{k_3 G'}{\Lambda} - \left(\frac{k_2 + k_0 G}{\Lambda}\right)^2 + k_0^2 - k_3^2\,. \tag{79}$$

- Case(1).
  In the limit $x_1 \to \infty$, if $\Lambda(x^1) = \infty$, we find $\omega^2 = k_0^2 - k_3^2$, and from this result, if $\omega^2 > 0$, we can write $\chi_1(x^1)$ as

$$\chi_1(x^1) = \exp\left\{\mathrm{i}\sqrt{k_0^2 - k_3^2}\,x^1\right\}. \tag{80}$$

- Case(2).
  When $x_1 \to \infty$, if $\Lambda(x^1) = b = const$, then we find $\omega^2 = k_0^2 - k_3^2 - [(k_2 + ak_0)/b]^2$, where a is a constant. From this result, if $\omega^2 > 0$, we can write

$$\chi_1(x^1) = \exp\left\{\mathrm{i}\sqrt{k_0^2 - k_3^2 - \left(\frac{k_2 + ak_0}{b}\right)^2}\,\xi\right\}. \tag{81}$$

- Case(3)
  Assuming $\xi \to 0$, than $\Lambda(x^1) = c = const$, and we find $\omega^2 = k_0^2 - k_3^2 - [(k_2 + dk_0)/c]^2$, where d is a constant. From this result, if $\omega^2 > 0$, we can write

$$\chi_1(x^1) = \exp\left\{\mathrm{i}\sqrt{k_0^2 - k_3^2 - \left(\frac{k_2 + dk_0}{c}\right)^2}\,x^1\right\}. \tag{82}$$

7. Results and discussion

In the present paper, we investigated the mDKP equation and the Maxwell equations in the background of the general rotating space-time.

(a) We showed that charge-less and massless spin-1 particle and free-space Maxwell equation satisfy the same equations.

(b) For each component of the mDKP spinor, the corresponding Maxwell field-strength tensor components are found.

(c) By using the mDKP equation, it is shown that the necessity of $(3 + 1)$ space-time splitting is not required for the electromagnetic fields.

(d) For two different well-known cosmological models, using our result, exact solutions are easily obtained.

(e) We find the oscillating regions of the massless spin-1 particles in three different limits.

This features strongly motivates us to use the mDKP equation to investigate the behaviour of the electromagnetic field. Another motivation is that the results obtained can be used to study quantum field theory in curved rotating space-times. Also, the wave functions obtained can be used to discuss the photon production in some special space-times which are included in the general rotating space-time that we defined in Sec. 2.

References

[1] H. Tetrode, Z. Phys. **50** (1928) 336.

[2] H. Weyl, Z. Phys. **56** (1929) 330.

[3] V. A. Fock, Z. Phys. **57** (1929) 261.

[4] V. A. Fock and D. Ivanenko, Compt. Rend. Acad. Sci. **188** (1929) 1470.

[5] V. A. Fock, Compt. Rend. Acad. Sci. **189** (1929) 25.

[6] W. L. Bade and H. Jehle, Rev. Mod. Phys. **25** (1953) 714.

[7] D. Brill and J. Wheeler, Rev. Mod. Phys. **29** (1957) 465.

[8] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York (2002).

[9] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, New York (1982).

[10] M. Gasperini and V. de Sabbata, *Introduction to Gravitation*, World Scientic (1985).

[11] R. H. Good, Phys. Rev. **5** (1957) 1914.

[12] H. E. Moses, Nuovo Cim. Suppl. **7** (1958) 1; Phys. Rev. **113** (1959) 1670.

[13] G. Petiau, Acad. R. Belg. Cl. Sci. Mem. Collet. (8) **16**, No. 2 (1936); R. J. Duffin, Phys. Rev. **54** (1938) 1114; N. Kemmer, Proc. Roy. Soc. (London) A **173** (1939) 91; N. Kemmer, Proc. Camb. Phil. Soc. **39** (1943) 189; M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Japan **22** (1940) 757 [Suppl. Progr. Theor. Phys. **22** (1955) 84].

[14] R. A. Krajcik and M. M. Nieto, Am. J. Phys. **45** (1977) 818.

[15] V. Gribov, Eur. Phys. J. C **10** (1999) 71.

[16] I. V. Kanatchikov, Rept. Math. Phys. **46** (2000) 107.

[17] V. M. Red'kov, quant-ph/9812007.

[18] J. T. Lunardi, B. M. Pimentel and R. G. Teixera, in *Geometrical Aspects of Quantum Fields*, Proc. 2000 Londrina Workshop, Londrina, Brazil, eds. A. A. Bytsenko, A. E. Golcalves and B. M. Pimentel, World Scientic, Singapore (2001).

[19] N. Ünal, Found. Phys. **27** (1997) 795.

[20] N. Ünal and Y. Sucu, Int. J. Mod. Phys. A **17** (2002) 1137.

[21] A. Havare and T. Yetkin, Class. Quantum Grav. **19** (2002) 2783.

[22] M. Salti and A. Havare, Mod. Phys. Lett. A **20** (2005) 451.

[23] R. X. Saibabalov, Gen. Rel. Grav. **27** (1995) 697.

[24] A. Havare and T. Yetkin, Chin. J. Phys. **41** No. 5 (2003).

[25] H. V. Fagundes, Gen. Rel. Cos. **24** No. 2 (1992).

[26] L. C. Garcia de Andrade, Gen. Rel. Grav. **35** (2003) 1279.

[27] R. M. Avakyan, E. V. Chubaryan and A. H. Yeranyan, gr-qc/0102030.

[28] G. Gamow, Nature (London) **158** (1946) 549.

[29] K. Gödel, Rev. Mod. Phys. **21** (1949) 447.

[30] O. Gron and H. H. Soleng, Acta Physica Polonica B **20** (1989) 550.

BEZMASENE ČESTICE SPINA 1 U ROTIRAJUĆEM PROSTORU-VREMENU

Razmatramo bezmasenu jednadžbu Duffin–Kemmer–Petiau-a za opći slučaj rotirajućeg prostora-vremena i nalazimo njen oblik drugog reda za pojedinu geometriju. Primjenom te jednadžbe drugog reda postigli smo egzaktna rješenja Duffin–Kemmer–Petiau-ve jednadžbe za dva poznata kozmološka modela. Pored toga, primjenom Maxwellovih jednadžbi u spinornom obliku, problem širenja valova svodi se na rješavanje diferencijalne jednadžbe drugog reda za kompleksni slog električnih i magnetskih polja. Tim dvama pristupima dobivamo spinore izražene preko tenzora jakosti polja.