Printed ISSN 1330–0016
Online ISSN 1333–9133
CD ISSN 1333–8390
CODEN FIZBE7

INTERPRETATIONS OF OCTONION WAVE EQUATIONS

P. S. BISHT[a], B. PANDEY[a] and O. P. S. NEGI[b,1]

[a]*Department of Physics, Kumaun University, S. S. J. Campus, Almora - 263601 (Uttarakhand), India*

[b]*Institute of Theoretical Physics, Chinese Academy of Sciences, KITP Building Room No.- 6304, Hai Dian Qu Zhong Guan Chun Dong Lu, 55 Hao, Beijing - 100080, P. R. China*

E-mail addresses: ps_bisht123@rediffmail.com, ops_negi@yahoo.co.in

Received 13 August 2007; Revised manuscript received 12 August 2008
Accepted 17 September 2008 Online 30 October 2008

The interpretations of octonion wave equations in eight-dimensional space-time is studied. We made an attempt to consider the octonion field equation as the equation of motion for particles carrying simultaneously electric and magnetic charges (i.e. dyons) in external and internal spaces. It has been concluded that the component of octonion potential wave equations behaves neither as the generalized electromagnetic fields of monopoles nor of the dyons. Rather, it has a mixed behaviour of electromagnetic fields associated with the electric and magnetic charges in external and internal spaces. We have also made an attempt to investigate the split octonion wave equation and its interpretation in classical electrodynamics, and accordingly the consistent and compact forms of eight-dimensional potential and current equation of dyons are obtained in terms of Zorn's vector matrix realization of split octonions. Visualizing the external four-space as the localization space for tachyons, it is shown that the split octonion wave equation reduces to the Maxwell's equation (field equation) for bradyons in R^4-space as well as that for tachyons in T^4- space in the absence of other.

PACS numbers: 02.30.-f, 03.65.Fd UDC 530.145

Keywords: octonion wave equation, eight-dimensional space-time, dyons, Zorn's vector matrix, bradyons, tachyons

1. Introduction

Octonions were first introduced in physics by Jordan, von Neuman and Wigner [1], who investigated a new finite Hilbert space, on replacing the com-

[1]Permanent address- Department of Physics, Kumaun University, S. S. J. Campus, Almora -263601 (U.A.) India

plex numbers by octonions [2]. According to the celebrated Hurwitz theorem [3], there exist four-division algebras consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathcal{O}$ (octonions). All four algebras are alternative with antisymmetric associators. In 1961, Pais [4] pointed out a striking similarity between the algebra of interactions and the split octonion algebra and, accordingly, some attention has been directed to octonions in theoretical physics with the hope of extending the $3+1$ space-time framework of the theory to eight dimensions to accommodate the ever increasing quantum numbers and internal symmetries assigned to elementary particles and gauge fields. Much literature is available [5 − 11] on the applications of octonions to interpret the wave equation, Dirac equation, and the extension of octonion non-associativity to physical theories. It is assumed that the octonion wave equation contains four-dimensional internal space of mass parameters, while the external four-dimensional space has been interpreted as the four generators of fermions within the quark-lepton symmetry. In this direction, the ingenious work was done by Günaydin and Gürsey [12] to formulate quark models and colour gauge theory in terms of split octonions. The $SU(3)$ group appears as the automorphism group of octonion representation, leaving the complex subspace and the scalar product invariant. This approach has been extended by many [13 − 18] to investigate the role of octonions and division algebra in unified gauge theories, higher-dimensional theories of supersymmetry and super-strings. Octonions were also used by Buoncristiani [19] in writing Yang-Mill's field equation in a simpler form. The extension of quaternion matrices to octonions for their interpretation in non-Riemannian geometry has been analysed by Marques et al. [20].

In the past few years, there has been a considerable interest in higher-dimensional kinematical models [21] for a proper and unified representation of a relativistic object, bradyonic as well as tachyonic (including those with internal structure). It has been speculated [22] that the problem of representation and localization of tachyonic objects may be solved only with the extension of four-dimensional Minkowski space to higher-dimensional space-time. It has already been discussed in a series of papers [23 − 25] that the true localization space for the representation of tachyons is T^4-space with one space and three time coordinates, while that for bradyons is the usual R^4-space with one time and three space coordinates. The unified eight-dimensional space time has been discussed earlier [25] as the unified space of bradyons and tachyons, i.e. $R^8 = R^4 \bigcup T^4$. The two R^4- and T^4-localization spaces are considered as external (internal) space for bradyons (tachyons) and vice versa. Moreover, the built-in duality associated with the combination of symmetries of R^4- and T^4-subspaces is useful to understand the problem of quark confinement in quantum chromodynamics where the role of tricolors would be played by three time coordinates for bradyons and that of three space coordinates for tachyons.

In order to interpret octonion wave equations in eight-dimensional space-time, we made in the present paper an attempt to discuss the octonion field equation as the equation of motion for particles carrying simultaneously electric and magnetic (monopole [26]) charges (i.e. dyons [27]). Starting from the regularity condition of octonion field equation, we have developed a consistent and compact formulation of eight-dimensional potential and current equations for dyons. It has been demonstrated that the component of octonion potential wave equations behaves neither as the generalized electromagnetic fields of monopoles nor as the dyons. Rather, they have the mixed behaviour of electromagnetic fields associated with the electric and magnetic charges in external and internal spaces.

After decomposing the octonion wave equation into two quaternion-valued wave equations, it is shown that the two different spaces describe two separate wave equations for electromagnetic fields. Visualizing the external four-spaces as the localization space for bradyons and the internal space as the localization space for tachyons, it is shown that the octonion wave equation reduces to the Maxwell's equation (field equation) for bradyons in R^4-space as well as that for tachyons in T^4-space. It has also been emphasized that the quaternionic decomposition of octonion wave equation describes the field equations of dyons and tachyonic dyons, respectively, when we take them as the combination of bi-quaternion instead of real quaternion in external and internal four-dimensional spaces of eight-dimensional space-time.

Then we have made an attempt to investigate the split octonion wave equation and its interpretation in classical electrodynamics. Split octonion electrodynamics has been discussed in terms of Zorn's vector matrix realization by describing electrodynamics potential, current and other dynamical quantities as octonion variables. Also, the consistent and compact forms of eight-dimensional potential and current equation of dyons are obtained in terms of Zorn's vector matrix realization of split octonions. It has been shown that the split octonion valued potential wave equation also behaves neither as the generalized electromagnetic fields of monopoles nor of the dyons. Rather, it has again a mixed behaviour of electromagnetic fields associated with the electric and magnetic charges in external and internal spaces. And last, it is shown that the split octonion Zorn's vector realization reproduces two different spaces to demonstrate the separate wave equations for electromagnetic fields. Visualizing the external four-space as the localization space for tachyons, it is shown that the octonion wave equation, when expressed in terms of split octonions, reduces to the Maxwell's equation (field equation) for bradyons in R^4-space as well as that for tachyons in T^4-space in the absence of other.

2. *Definition of octonion*

An octonion x is expressed as a set of eight real numbers

$$x = e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 = e_0x_0 + \sum_{A=1}^{7} e_A x_A \, , \quad (1)$$

where $e_A (A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. Set of octets $(e_0, \, e_1, \, e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and satisfy the following multiplication rules

$$e_0 = 1; \; e_0e_A = e_Ae_0 = e_A; \; e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C. \; (A, B, C = 1, 2, \ldots, 7) \quad (2)$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 for following combinations,

$$f_{ABC} = \quad +1; \quad \forall (ABC) = (123), (471), (257), (165), (624), (543), (736). \quad (3)$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space of dimension 8. As such, we may write the following relations among octonion basis elements

$$\begin{aligned} [e_A,\ e_B] &= 2\, f_{ABC}\, e_C, \\ \{e_A,\ e_B\} &= -2\, \delta_{AB} e_0\, , \\ e_A(\, e_B\, e_C) &\neq (e_A\, e_B\,)\, e_C\, , \end{aligned} \qquad (4)$$

where brackets [,] and { , } are used respectively for commutation and the anti-commutation relations, while δ_{AB} is the usual Kronecker delta-Dirac symbol. Octonion conjugate is defined as

$$\overline{x} = e_0x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7 = e_0x_0 - \sum_{A=1}^{7} e_A x_A\, , \quad (5)$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence, an octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$Sc(x) \quad = \quad \frac{1}{2}(\, x \, + \, \overline{x}\,); \quad Vec(x) = \frac{1}{2}(\, x \, - \, \overline{x}\,) = \sum_{A=1}^{7} e_A x_A\, . \qquad (6)$$

Conjugates of product of two octonions and its own are described as

$$\overline{(x\,y)} = \quad \overline{y}\, \overline{x}\, , \qquad \overline{(\overline{x})} = x\, . \qquad (7)$$

while the scalar product of two octonions is defined as

$$\langle x\, ,\, y\rangle \quad = \tfrac{1}{2}(x\,\overline{y} + y\,\overline{x}) = \tfrac{1}{2}(\overline{x}\, y + \overline{y}\, x) = \quad \sum_{\alpha=0}^{7} x_\alpha\, y_\alpha\, . \qquad (8)$$

The norm $N(x)$ and inverse x^{-1}(for a nonzero x) of an octonion are respectively defined as

$$\begin{aligned} N(x) = x\,\overline{x} = \overline{x}\, x \quad &= \quad \sum_{\alpha=0}^{7} x_\alpha^2 . e_0\, , \\ x^{-1} \quad &= \quad \frac{\overline{x}}{N(x)} \Longrightarrow x\, x^{-1} = x^{-1}\, x = 1\, . \end{aligned} \qquad (9)$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(x\,y) = \quad N(x)\,N(y) = \quad N(y)\,N(x)\,. \tag{10}$$

Equation (4) shows that octonions are not associative in nature and thus do not form the group in their usual form. Non-associativity of octonion algebra $\mathcal{O}$ is provided by the associator $(x, y, z) = (xy)z - x(yz)\ \forall x, y, z \in \mathcal{O}$ defined for any 3 octonions. If the associator is totally antisymmetric for exchanges of any 2 variables, i.e. $(x, y, z) = -(z, y, x) = -(y, x, z) = -(x, z, y)$, then the algebra is called alternative. Hence, the octonion algebra is neither commutative nor associative but, it is alternative.

3. *Octonion wave equation*

In order to write the octonion wave equation, let us define the differential octonion D as

$$D \quad = \quad \sum_{\mu=0}^{7} e_\mu D_\mu\,, \tag{11}$$

where D_μ are described as the components of differential operator in eight-dimensional representation. Here we assume the eight-dimensional space as the combination of two (external and internal) four-dimensional spaces. As such, we describe a function of octonion variable as

$$\mathcal{F}(X) \quad = \textstyle\sum_{\mu=0}^{7} \quad e_\mu f_\mu(X) = f_0 + e_1 f_1 + e_2 f_2 + \ldots.. + e_7 f_7\,, \tag{12}$$

where f_μ are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions. As such, a function $\mathcal{F}(X)$ of an octonion variable $X = \sum_{\mu=0}^{7} e_\mu X_\mu$ is left regular at X if and only if $\mathcal{F}(X)$ satisfies the condition

$$D\mathcal{F}(X) \quad = \quad 0\,. \tag{13}$$

Similarly, a function $G(X)$ is a right regular if and only if

$$G(X)\overleftarrow{D} \quad = \quad 0\,, \tag{14}$$

where $G(X) = g_0 + g_1 e_1 + g_2 e_2 + \ldots.. + g_7 e_7$. Then we get

$$D\mathcal{F} \quad = \quad I_0 + I_1 e_1 + I_2 e_2 + I_3 e_3 + I_4 e_4 + I_5 e_5 + I_6 e_6 + I_7 e_7\,, \tag{15}$$

where

$$\begin{aligned}
I_0 &= \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7 \,, \\
I_1 &= \partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 + \partial_6 f_5 - \partial_5 f_6 - \partial_7 f_4 + \partial_4 f_7 \,, \\
I_2 &= \partial_0 f_2 + \partial_2 f_0 + \partial_3 f_1 - \partial_1 f_3 + \partial_4 f_6 - \partial_6 f_4 - \partial_7 f_5 + \partial_5 f_7 \,, \\
I_3 &= \partial_0 f_3 + \partial_3 f_0 + \partial_1 f_2 - \partial_2 f_1 + \partial_6 f_7 - \partial_7 f_6 + \partial_5 f_4 - \partial_4 f_5 \,, \\
I_4 &= \partial_0 f_4 + \partial_4 f_0 + \partial_3 f_5 - \partial_5 f_3 - \partial_2 f_6 + \partial_6 f_2 - \partial_1 f_7 + \partial_7 f_1 \,, \\
I_5 &= \partial_0 f_5 + \partial_5 f_0 + \partial_1 f_6 - \partial_6 f_1 + \partial_7 f_2 - \partial_2 f_7 - \partial_3 f_4 + \partial_4 f_3 \,, \\
I_6 &= \partial_0 f_6 + \partial_6 f_0 - \partial_1 f_5 + \partial_5 f_1 + \partial_2 f_4 - \partial_4 f_2 - \partial_3 f_7 + \partial_7 f_3 \,, \\
I_7 &= \partial_0 f_7 + \partial_7 f_0 + \partial_1 f_4 - \partial_4 f_1 + \partial_2 f_5 - \partial_5 f_2 - \partial_6 f_3 + \partial_3 f_6 \,. \qquad (16)
\end{aligned}$$

The regularity condition (13) may now be considered as homogeneous octonion wave equation for octonion variables without sources. On the other hand, equation (15) is considered as the inhomogeneous wave equation

$$D\mathcal{F} \;=\; I \,, \qquad (17)$$

where I is again an octonion. Similarly, we may also write the homogeneous as well as inhomogeneous octonion wave equations on using the right regularity condition (14). We may now interpret these octonion wave equations as the classical wave (field) equations of physical variables.

3.1. Potential equation for electromagnetic fields with sources

Let us consider the case of generalized electromagnetic fields of dyons (particles carrying simultaneously the electric and magnetic charges). We may now define an octonion valued potential, in eight dimensional formalism as the combinations of two four-dimensional spaces, as follows

$$\emptyset \;= \textstyle\sum_{\kappa=0}^{7} \quad e_\kappa \emptyset_\kappa = \displaystyle\sum_{\mu=0}^{3} e_\mu A_\mu + \sum_{\nu=4}^{7} e_\nu B_\nu \,, \qquad (18)$$

where $\emptyset$ is the octonion potential and A_μ and B_ν are assumed respectively as the electric and magnetic four-potentials associated with the electric and magnetic charges of dyons [28]. As such, we may write the wave equation for octonion potential variable for simultaneous existence of electric and magnetic charges on a particle (namely dyons) in the following manner,

$$\overline{D}\emptyset \;=\; F \,, \qquad (19)$$

where

$$\overline{D} \;=\; \sum_{\mu=0}^{7} \overline{e_\mu}\, D_\mu = e_0 D_0 - \sum_{A=1}^{7} e_A D_A \,, \text{ and } F = \sum_{\mu=0}^{7} F_\mu e_\mu \,. \qquad (20)$$

The coefficients of octonion F are then given by

$$\begin{aligned}
F_0 &= \partial_0\emptyset_0 + \partial_1\emptyset_1 + \partial_2\emptyset_2 + \partial_3\emptyset_3 + \partial_4\emptyset_4 + \partial_5\emptyset_5 + \partial_6\emptyset_6 + \partial_7\emptyset_7\,, \\
F_1 &= \partial_0\emptyset_1 - \partial_1\emptyset_0 + \partial_3\emptyset_2 - \partial_2\emptyset_3 + \partial_7\emptyset_4 - \partial_4\emptyset_7 + \partial_5\emptyset_6 - \partial_6\emptyset_5\,, \\
F_2 &= \partial_0\emptyset_2 - \partial_2\emptyset_0 + \partial_1\emptyset_3 - \partial_3\emptyset_1 + \partial_6\emptyset_4 - \partial_4\emptyset_6 + \partial_7\emptyset_5 - \partial_5\emptyset_7\,, \\
F_3 &= \partial_0\emptyset_3 - \partial_3\emptyset_0 + \partial_2\emptyset_1 - \partial_1\emptyset_2 + \partial_4\emptyset_5 - \partial_5\emptyset_4 + \partial_7\emptyset_6 - \partial_6\emptyset_7\,, \\
F_4 &= \partial_0\emptyset_4 - \partial_4\emptyset_0 - \partial_3\emptyset_5 + \partial_5\emptyset_3 + \partial_2\emptyset_6 - \partial_6\emptyset_2 + \partial_1\emptyset_7 - \partial_7\emptyset_1\,, \\
F_5 &= \partial_0\emptyset_5 - \partial_5\emptyset_0 - \partial_1\emptyset_6 + \partial_6\emptyset_1 - \partial_7\emptyset_2 + \partial_2\emptyset_7 + \partial_3\emptyset_4 - \partial_4\emptyset_3\,, \\
F_6 &= \partial_0\emptyset_6 - \partial_6\emptyset_0 + \partial_1\emptyset_5 - \partial_5\emptyset_1 - \partial_2\emptyset_4 + \partial_4\emptyset_2 + \partial_3\emptyset_7 - \partial_7\emptyset_3\,, \\
F_7 &= \partial_0\emptyset_7 - \partial_7\emptyset_0 - \partial_1\emptyset_4 + \partial_4\emptyset_1 - \partial_2\emptyset_5 + \partial_5\emptyset_2 + \partial_6\emptyset_3 - \partial_3\emptyset_6\,.
\end{aligned} \tag{21}$$

We get $F_0 = 0$ in the Euclidean space $(+,+,+,+,+,+,+,+)$ due to eight-dimensional Lorentz gauge condition (i.e., the combination of two gauges associated with electric and magnetic potentials). Thus, one-dimensional octonion representation is identical to eight-dimensional space over the field of real numbers. It is isomorphic to four-dimensional space representation over the field of complex variables which is equivalent to two-dimensional space representation over quaternion field variables. Similarly, one-dimensional quaternion space is isomorphic to four-dimensional space over the field of real numbers which is identical to two-dimensional space over the field of complex numbers. So, an octonionic potential may also be described as the combination of two quaternion potentials in the following manner

$$\emptyset = \emptyset_a + e_7\emptyset_b\,, \quad D = D_a + e_7 D_b\,, \quad F = F_a + e_7 F_b\,, \tag{22}$$

where $\emptyset_a$, $\emptyset_b$, D_a, D_b, F_a and F_b are quaternion variables described as

$$\begin{aligned}
\emptyset_a &= \emptyset_0 + e_1\emptyset_1 + e_2\emptyset_2 + e_3\emptyset_3\,, \\
\emptyset_b &= \emptyset_7 + e_1\emptyset_4 + e_2\emptyset_5 + e_3\emptyset_6\,, \\
D_a &= \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3\,, \\
D_b &= \partial_7 + e_1\partial_4 + e_2\partial_5 + e_3\partial_6\,, \\
F_a &= F_0 + e_1F_1 + e_2F_2 + e_3F_3\,, \\
F_b &= F_7 + e_1F_4 + e_2F_5 + e_3F_6\,.
\end{aligned} \tag{23}$$

Here e_1, e_2 and e_3 are three quaternion units which satisfy the multiplication rule $e_j e_k = -\delta_{jk} + \varepsilon_{jkl} e_l \quad \forall (j,k,l = 1,2,3)$. If we replace the octonion unit e_7 by imaginary unit $\mathrm{i} = \sqrt{-1}$, which commutes with other octonion basis elements e_A, the resultant theory of octonions becomes the theory of bi-quaternion variables and, accordingly, the generalized fields of dyons are already written in compact, covariant and consistent way [28]. We may now consider the following mapping for different four-dimensional spaces,

$$\begin{aligned} &\partial_7 \to \partial_0^{'}, \quad \emptyset_7 \to B_0, \quad \emptyset_0 \to A_0\,, \\ &\partial_4 \to \partial_1^{'}, \quad \emptyset_4 \to B_1, \quad \emptyset_1 \to A_1\,, \\ &\partial_5 \to \partial_2^{'}, \quad \emptyset_5 \to B_2, \quad \emptyset_2 \to A_2\,, \\ &\partial_6 \to \partial_3^{'}, \quad \emptyset_6 \to B_3, \quad \emptyset_3 \to A_3\,. \end{aligned} \tag{24}$$

Then we get

$$\begin{aligned} F_j &= \partial_0 A_j - \partial_j A_0 + (\overrightarrow{\nabla} \times \overrightarrow{A})_j + \partial_0^{'} B_j - \partial_j^{'} B_0 + (\overrightarrow{\nabla} \times \overrightarrow{B})_j\,, \\ F_{j+3} &= \partial_0 B_j - \partial_j B_0 + (\overrightarrow{\nabla} \times \overrightarrow{B})_j + \partial_0^{'} A_j - \partial_j^{'} A_0 + (\overrightarrow{\nabla}^{'} \times \overrightarrow{A})_j\,, \\ F_7 &= \partial_0 B_0 - \partial_1 B_1 - \partial_2 B_2 - \partial_3 B_3 - (\partial_0^{'} A_0 + \partial_1^{'} A_1 + \partial_2^{'} A_2 + \partial_3^{'} A_3)\,. \end{aligned} \tag{25}$$

Hence Eq. (25) shows that F_0, F_j, F_{j+3} and F_7 correspond to the components of electric and magnetic fields obtained in terms of components of two four-potentials (A_μ and B_μ) in internal and external spaces. If these two spaces are completely disjoint spaces, we get only F_j, and F_0 is vanishing due to Lorentz gauge condition, while F_{j+3} and F_7 do not occur. Here prime derivatives are associated with the second part of four-dimensional structure of eight-potential. In Eq. (25), we find that F_j is made up of the j^{th} components of electric and magnetic fields. Here the first term $\partial_0 A_j - \partial_j A_0$ describes the electric field and second term $(\overrightarrow{\nabla} \times \overrightarrow{A})_j$ corresponds to the magnetic field in usual four-dimensional (we call it as the external) space. In the light of the duality of electric and magnetic fields (and accordingly for electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four-potentials), the third term $\partial_0 B_j - \partial_j B_0$ of F_j is equivalent to the magnetic field while the last term $(\overrightarrow{\nabla} \times \overrightarrow{B})_j$ denotes the electric field in other four-dimensional (let us call it as internal or magnetic) space. On mixing of these two spaces, $\partial_0 A_j - \partial_j A_0 + (\overrightarrow{\nabla} \times \overrightarrow{A})_j$ resembles with the expression of generalized electric field and $\partial_0^{'} B_j - \partial_j^{'} B_0 + (\overrightarrow{\nabla} \times \overrightarrow{B})_j$ as that of the generalized magnetic fields of dyons [28, 29]. So, it looks awkward to combine external (internal) space of electric charge (magnetic monopole) to the internal (external) space of magnetic monopole (electric charge) to interpret F_j. As such, we may conclude that the Eq. (25) does not represent the true generalization of potential equation for the generalized electromagnetic fields of dyons. Rather, we have obtained the mixed behaviour of electric and magnetic charges in internal and external spaces. So, octonion wave equation of electromagnetic potential faces difficulties and hence needs modification to interpret it consistently.

3.2. Current equation

We now analize the octonion wave equation (17) as the current equation in eight-dimensional representations, i.e.

$$D \quad F = \quad S\,, \tag{26}$$

where

$$\begin{aligned}
S_0 &= \partial_0 F_0 - \partial_1 F_1 - \partial_2 F_2 - \partial_3 F_3 - (\partial_1^{'} F_4 + \partial_2^{'} F_5 + \partial_3^{'} F_6 + \partial_0^{'} F_7)\,, \\
S_j &= \partial_0 F_j + \partial_j F_0 + (\overrightarrow{\nabla} \times \overrightarrow{F})_j - \partial_0^{'} F_{j+3} + \partial_j^{'} F_7 - (\overrightarrow{\nabla}' \times \overrightarrow{F})_{j+3}\,, \\
S_{j+3} &= \partial_0^{'} F_j + \partial_j^{'} F_0 - (\overrightarrow{\nabla}' \times \overrightarrow{F})_j - \partial_0 F_{j+3} - \partial_j F_7 - (\overrightarrow{\nabla} \times \overrightarrow{F})_{j+3}\,, \\
S_7 &= \partial_1 F_4 + \partial_2 F_5 + \partial_3 F_6 + \partial_0 F_7 + (\partial_0^{'} F_0 - \partial_1^{'} F_1 - \partial_2^{'} F_2 - \partial_3^{'} F_3)\,. \quad (27)
\end{aligned}$$

Thus, the current equation for octonion variables describes [30] the generalized structure of differential equations (Maxwell's equations) in eight-dimensional space-time. Four-dimensional reduction of these differential equations may be visualized as the Maxwell's equations in internal and external spaces. So, it is difficult to explain the various terms associated with the eight parameters $S_\Lambda(\Lambda = 0, 1, 2, 3..., 7)$ correctly. Silagade [30] provided the interpretation of the homogeneous octonion wave equation $DF = 0$ as equivalent to one of the pair of seven-dimensional Maxwell's equations and the second pair of seven-dimensional Maxwell's equations may be obtained on applying the duality transformations between electric and magnetic fields therein. Similarly, Gamba and Gogberashvili [8] also tried to explain the structure of eight parameters $S_\Lambda(\Lambda = 0, 1, 2, 3..., 7)$ but the consistent justification needs modifications. Hence, octonion wave equation (26) may be identified rather a current equation or octonion field equation in eight-dimensional spaces for mixed structural behaviour of sources (the electric and magnetic) instead of dyons. In other words, we may say that Eq. (26) can not be described as the true field equation of dyons. Here S_0 and S_7 may be taken to vanish due to the equations of continuity in internal and external spaces.

3.3. Quaternion decomposition

We may decompose an octonion in terms of two-quaternions. So, let us decompose octonion wave equation for potential in terms of two quaternions as

$$\overline{D}\emptyset = (\overline{D}_a - e_7 D_b)(\emptyset_a + e_7 \emptyset_b) \quad = \quad (\phi + e_7 \varphi) = F\,. \tag{28}$$

Here

$$\phi = \overline{D}_a \emptyset_a + \emptyset_b \widetilde{D}_b = (D_0 - D_1 e_1 - D_2 e_2 - D_3 e_3)\emptyset_a + \emptyset_b(\overleftarrow{D_7} - \overleftarrow{D_4} e_1 - \overleftarrow{D_5} e_2 - \overleftarrow{D_6} e_3)\,, \tag{29}$$

where $\overleftarrow{D_r}, (r = 4, 5, 6, 7)$ represents the partial differential of $\emptyset_b$ from right to left, $\widetilde{D}_b$ denotes the quaternion conjugation ($\tilde{e_o} = e_0\,, \quad \tilde{e_j} = -e_j \quad \forall j = 1, 2, 3$) from right to left, i.e.,

$$\overline{D}_a = D_0 - D_1 e_1 - D_2 e_2 - D_3 e_3\,,$$

$$\overline{\widetilde{D}_a} = D_0 + D_1 e_1 + D_2 e_2 + D_3 e_3 = D_1\,,$$

$$\widetilde{D}_b = D_7 - D_4 e_1 - D_5 e_2 - D_6 e_3$$

and
$$\varphi = \overline{\widetilde{D}}_a \emptyset_b - \emptyset_a D_b = D_a \emptyset_b - \emptyset_a D_b \,. \tag{30}$$

Here an octonion-valued potential is described as the sum of two quaternions by the Cayley-Dickson doubling process. Thus, we observe that the octonion potential wave equation rules out the existence of isolated magnetic monopole and provides a mixed structural behaviour of electromagnetic fields containing both charges simultaneously. In other words, we may say that it is a generalized field equation of electromagnetic fields associated with a particle carrying simultaneously electric and magnetic charges (namely dyons). The generalized electromagnetic fields of dyons are symmetric and dual invariant in terms of two four potentials. So, the possibility of monopoles (or dyons) in non-Abelian or supersymmetric gauge theories is directly linked with the existence of second four-potential which is hidden in external space and present in internal space, while the electric four-potential is the consequence of our external space and seems to be hidden in internal space.

Similarly we may decompose the current equation (26) in terms of two quaternions as

$$DF \quad = (D_a + e_7 D_b)(g + e_7 h) \quad = S = s_a + e_7 s_b \,, \tag{31}$$

where
$$s_a = D_a g - h D_b = D_a \overline{D}_b \emptyset_a + \emptyset_b D_b \overline{D}_b \,,$$

$$s_b = \widetilde{D}_a h + g D_b = \overline{D}_a D_a \emptyset_b + \emptyset_b \overline{D_b} D_b \,. \tag{32}$$

Thus $S = s_a + e_7 s_b$ is the octonionic form of generalized current density. Similarly, we may express the octonionic forms of generalized force and generalized field tensor density as the generalizations of two quaternions.

3.4. Localization spaces of bradions and tachions

We have described the octonion (eight-dimensional) space as made of two quaternion (namely the external and internal four-dimensional) spaces. Let us suppose the external four-space as the usual Minkowski (or Euclidean) $R^4 \Rightarrow M\,(1,3)$-space (consisting one time and three space coordinates). This space has been named [20, 21] as the localization space of bradyons (particles travelling slower than light, subluminal particles). Accordingly, we describe the internal four-dimensional space as the $T^4 \Rightarrow M\,(3,1)$-space (consisting three time and one space coordinates). The possibility of such space is explored [20, 21] as the localization space of tachyons (particles travelling faster than light, superluminal particles). Hence an octonion eight-dimensional space is described as the unified space containing both external $R^4 \Rightarrow M\,(1,3)$ and internal $T^4 \Rightarrow M\,(3,1)$ four-dimensional spaces. In other words, we may identify the octonion space as the unified localization space for the description of bradyons and tachyons. Therefore, the bradyons are the objects of our four-dimensional world for which the Cauchy's data are described in the plane $\{t = 0\}$, while tachyons are localized in time and their Cauchy's data lie in the plane $\{x = 0\}$. Hence, the eight-dimensional space of octonion variables described above is the unified localization space for bradyons and tachyons and hence can be expressed as $R^8 = R^4 \bigcup T^4$. The T^4 space is characterized as the hidden space for bradyons while R^4-space is identified as the hidden space for tachyons. We may also interpret that the R^4-space is visualized as the internal space for tachyons (and external space for bradyons) and correspondingly the T^4-space is the internal space for bradyons (and external space for tachyons). First quaternion variable of Eq. (31) maps to four-dimensional space for bradyons, i.e. $R^4 \simeq (t, \overrightarrow{r})$, while the second quaternion variable gives rise to four-dimensional space for tachyons, i.e. $T^4 \simeq (r, \overrightarrow{t})$. In the absence of the second quaternion, Eq. (32) reduces to

$$s_a = \quad D_a \overline{D}_a \emptyset_a \quad \Rightarrow (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2})\phi = \Box(R)\phi\,, \tag{33}$$

which resembles to Cauchy-Feuter equation of quaternion variables in $R^4 \simeq (t, \overrightarrow{r})$ space and thus describes the usual Maxwell's equations in electromagnetic fields in R^4-space. On the other hand, in the absence of the first quaternion, Eq. (32) reduces to

$$s_b = \emptyset_b \quad \overline{D}_b D_b \quad \Rightarrow \varphi(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t_x^2} - \frac{\partial^2}{\partial t_y^2} - \frac{\partial^2}{\partial t_z^2}) = \varphi\overleftarrow{\Box}(T)\,, \tag{34}$$

which also describes the Cauchy–Feuter equation of quaternion variables in $T^4 \simeq (r, \overrightarrow{t})$ space and coincides with the Maxwell's equations of superluminal photons in T^4-space. Equation (33) is the left regular while Eq. (34) is the right regular. Hence, the regularity conditions are changed for internal and external spaces. In the case of bi-quaternion, Eqs. (33) and (34) characterize, respectively, the equations of dyons for subluminal and superluminal electromagnetic fields. In general, the octonionic space describes the unified structure of subluminal and superluminal objects. As such, the octonionic eight-dimensional representation is the unified picture of bradyons and tachyons and the octonionic current equations thus reproduces two different kinds of Maxwell's equations in external and internal spaces in order to explain the simultaneous description of bradyons and tachyons. Here, we have developed the octonion wave equation and theory of octonion field variables in compact and simpler manner. So, in order to overcome the non-associativity, it is necessary to decompose them in terms of two quaternions isomorphic to two different four-dimensional spaces. The compact and simpler form of octonion field equation provides a unified model of the theories of subluminal and superluminal electromagnetic fields in view of the localizability of bradyons and tachyons in consistent manner.

4. Split octonion wave equation

The split octonions are a non associative extension of quaternions (or the split quaternions). They differ from the octonion in the signature of quadratic form.

The split octonions have a signature $(4,4)$ whereas the octonions have positive signature $(8,0)$. The Cayley algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{C}$ is visualized as the algebra of split octonions with basis elements $u_0 = \frac{1}{2}(e_0 + \mathrm{i}e_7)$, $u_0^\star = \frac{1}{2}(e_0 - \mathrm{i}e_7)$, $u_j = \frac{1}{2}(e_j + \mathrm{i}e_{j+3})$, $u_j^\star = \frac{1}{2}(e_j - \mathrm{i}e_{j+3})$ ($j = 1, 2, 3$; $\mathrm{i} = \sqrt{-1}$) as the bi-valued (or bi-dimensional) representations of quaternion units e_0, e_1, e_2, e_3. The split Cayley (octonion) algebra is thus expressed in terms of 2×2 Zorn's vector matrix components which are scalar and vector parts of a quaternion. As such, we may also write an arbitrary split octonion A in terms of the following 2×2 Zorn's vector matrix realizations [24] as

$$A = au_0^\star + bu_0 + x_i u_i^\star + y_i u_i = \begin{pmatrix} a & -\overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix}. \tag{35}$$

Split octonion conjugation of equation (35) is then described as

$$\overline{A} = au_0 + bu_0 * -x_i u_i * -y_i u_i = \begin{pmatrix} b & \overrightarrow{x} \\ -\overrightarrow{y} & a \end{pmatrix}. \tag{36}$$

The norm of A is defined as,

$$\overline{A}A = A\overline{A} = (ab + \overrightarrow{x}.\overrightarrow{y})\hat{1}\,, \tag{37}$$

where $\hat{1}$ is the identity element given by $\hat{1} = 1u_0 + 1u_0^\star$. Any four-vector A_μ (complex or real) can equivalently be written in the following Zorn's matrix realization as

$$Z(A) = \begin{pmatrix} x_4 & -\overrightarrow{x} \\ \overrightarrow{y} & y_4 \end{pmatrix} \tag{38}$$

and

$$Z(\overline{A}) = \begin{pmatrix} x_4 & \overrightarrow{x} \\ -\overrightarrow{y} & y_4 \end{pmatrix}. \tag{39}$$

In Eqs. (38) and (39), putting $\overrightarrow{x} = \overrightarrow{y}$, we get the equivalent matrix realization for a four-vector in bi-valued four-dimensional Euclidean space-time. In order to develop the the eight-dimensional split octonion kinematics, we start with the following definition of the split octonion differential operator D and its conjugate $\overline{D}$ in terms of 2×2 Zorn's matrix realizations,

$$D = \begin{pmatrix} \partial_4 & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & \partial_4' \end{pmatrix}, \tag{40}$$

$$\overline{D} = \begin{pmatrix} \partial_4 & \overrightarrow{\nabla} \\ -\overrightarrow{\nabla} & \partial_4' \end{pmatrix}, \tag{41}$$

with the assumption that the primed variables are represented in internal space, whereas the unprimed variables are defined in external four-dimensional space.

4.1. Potential equation

Let us express an octonion potential in the following form

$$\emptyset \quad = \quad A_4 u_0 * + B_4 u_0 + B_i u_i * + A_i u_i \,, \tag{42}$$

where $\{A_\mu\} = (A_4, A_j)$ and $\{B_\mu\} = (B_4, B_j)$ are respectively two four-potentials associated with electric and magnetic charges. Equation (42) may then be expressed in terms of the split octonion 2×2 Zorn's vector matrix realization as

$$\emptyset \quad = \quad \begin{pmatrix} A_4 & -\overrightarrow{B} \\ \overrightarrow{A} & B_4 \end{pmatrix}, \tag{43}$$

where $A_4 = \mathrm{i}\,\emptyset_e$($e$ denotes the electric charge) and $B_4 = \mathrm{i}\,\emptyset_g$($g$ denotes the magnetic charge). Here $\overrightarrow{A}$ and A_4 are assumed to be the components of electric four-potential $A_\mu = (\overrightarrow{A}, \mathrm{i}\,\emptyset_e)$. Similarly $\overrightarrow{B}$ and B_4 are considered as the components of magnetic four-potential $B_\mu = (\overrightarrow{B}, \mathrm{i}\,\emptyset_g)$. We have designed the eight-dimensional space spanned by split octonion basis elements in terms of two four-dimensional spaces (namely the external and internal space). Here also the electric four-potential is described in external four-dimensional space, while the magnetic four-potential has been considered in internal four-dimensional spaces. Since electric charge and magnetic monopoles are dual to each other, we may interpret these two spaces as dual to each other. The internal space of electric charge may be identified as the external space of monopole while the external space of electric charge will be visualized as the internal space of magnetic monopole. Accordingly, the electric charge is considered in external space and magnetic charge in internal space. Hence, the eight-dimensional spaces have the built-in duality, where a bradyonic monopole plays the role of tachyonic electric charge while the tachyonic monopole looks like a bradyonic electric charge or vice versa. Eight-dimensional spaces have been considered as the unification of R^4-(bradyonic) and T^4-(tachyonic) spaces.

Let us now write the the inhomogeneous octonion wave equation in its split form as

$$\overline{D}\emptyset \;=\; \begin{pmatrix} \partial_4 A_4 + \overrightarrow{\nabla}.\overrightarrow{A} & -(\partial_4 \overrightarrow{B} - \overrightarrow{\nabla} B_4 - \overrightarrow{\nabla} \times \overrightarrow{A}) \\ \partial_4' \overrightarrow{A} - \overrightarrow{\nabla}' A_4 - \overrightarrow{\nabla}' \times \overrightarrow{B} & \partial_4' B_4 + \overrightarrow{\nabla}'.\overrightarrow{B} \end{pmatrix} = F \,, \tag{44}$$

where F is a field like the electromagnetic field in split octonion 2×2 Zorn's vector matrix realization as

$$F \quad = \quad \begin{pmatrix} f_4 & -\overrightarrow{f} \\ \overrightarrow{f}\,' & f_4' \end{pmatrix}. \tag{45}$$

Comparing Eqs. (44) and (45), we get

$$f_4 \quad = \quad \partial_4 A_4 + \overrightarrow{\nabla}.\overrightarrow{A} = 0 \,, \tag{46}$$

$$f'_4 \quad = \quad \partial'_4 B_4 + \overrightarrow{\nabla}'.\overrightarrow{B} = 0\,, \tag{47}$$

$$\overrightarrow{f} \quad = \quad \partial_4 \overrightarrow{B} - \overrightarrow{\nabla} B_4 - \overrightarrow{\nabla} \times \overrightarrow{A}\,, \tag{48}$$

$$\overrightarrow{f}' \quad = \quad \partial'_4 \overrightarrow{A} - \overrightarrow{\nabla}' A_4 - \overrightarrow{\nabla}' \times \overrightarrow{B}\,. \tag{49}$$

In Eqs. (46) and (47), we could get $f_4 = 0$ and $f'_4 = 0$ on applying Lorentz gauge conditions on electric and magnetic four-potentials. We now discuss the different cases in the following subsections.

(i) Bradyonic case

We may describe the theory of bradyons by substituting $B_4 = A_4$, $\overrightarrow{B} = \overrightarrow{A}$ in Eqs. (40) to (49) and correspondingly $\partial_4 = \partial'_4$, $\overrightarrow{\nabla} = \overrightarrow{\nabla}' = \partial$. Then we get

$$\overline{D}\emptyset \;= F \;= \begin{pmatrix} \partial_4 A_4 + \overrightarrow{\nabla}.\overrightarrow{A} & -(\partial_4 \overrightarrow{A} - \overrightarrow{\nabla} A_4 - \overrightarrow{\nabla} \times \overrightarrow{A}) \\ \partial_4 \overrightarrow{A} - \overrightarrow{\nabla} A_4 - \overrightarrow{\nabla} \times \overrightarrow{A} & \partial_4 A_4 + \overrightarrow{\nabla}.\overrightarrow{A} \end{pmatrix}. \tag{50}$$

Using the definition of electric and magnetic fields given by

$$\overrightarrow{E} \quad = -\tfrac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\emptyset_e \quad = -\frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\phi_e\,, \tag{51}$$

$$\overrightarrow{\mathcal{H}} \quad = \quad \overrightarrow{\nabla} \times \overrightarrow{A}\,, \tag{52}$$

and imposing Lorentz-gauge condition $\partial_4 A_4 + \overrightarrow{\nabla}.\overrightarrow{A} = \frac{\partial \emptyset_e}{\partial t} + \overrightarrow{\nabla}.\overrightarrow{A} = 0$, and using $\overrightarrow{\psi} = \overrightarrow{E} - \mathrm{i}\overrightarrow{\mathcal{H}}(i = \sqrt{-1})$, we get

$$-\partial_4 \overrightarrow{A} + \overrightarrow{\nabla} A_4 + \overrightarrow{\nabla} \times \overrightarrow{A} \quad = \quad -\mathrm{i}\overrightarrow{E} + \overrightarrow{\mathcal{H}} = -\mathrm{i}\overrightarrow{\psi}*\,, \tag{53}$$

$$\partial_4 \overrightarrow{A} - \overrightarrow{\nabla} A_4 - \overrightarrow{\nabla} \times \overrightarrow{A} \quad = \quad \mathrm{i}\overrightarrow{E} - \overrightarrow{\mathcal{H}} = \mathrm{i}\overrightarrow{\psi}*\,. \tag{54}$$

Thus Eq. (50) reduces to

$$\overline{D}\emptyset \quad = \quad F\,, \tag{55}$$

where

$$F \quad = \quad \begin{pmatrix} 0 & -\overrightarrow{F} \\ \overrightarrow{F} & 0 \end{pmatrix} = \mathrm{i}\psi*\,. \tag{56}$$

Equation (56) is the split octonion form of field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ describing the electric and magnetic fields. Off-diagonal vector components of split octonions (50) are the conjugates to each other and the scalar components along principal diagonal are associated with the quaternion scalars (i.e. real quaternion). Similarly, we get

$$D\overline{\emptyset} \quad = \quad F^T, \tag{57}$$

where

$$F^T \quad = \quad \begin{pmatrix} 0 & \overrightarrow{F}* \\ -\overrightarrow{F}* & 0 \end{pmatrix} = \mathrm{i}\psi\,. \tag{58}$$

and Eqs. (55) and (57) lead to

$$\frac{1}{2}(\overline{D}\emptyset + D\overline{\emptyset}) \quad = \quad \begin{pmatrix} 0 & \overrightarrow{\mathcal{H}} \\ -\overrightarrow{\mathcal{H}} & 0 \end{pmatrix} = \overrightarrow{\mathcal{H}}\,, \tag{59}$$

$$\frac{1}{2}\mathrm{i}\,(\overline{D}\emptyset - D\overline{\emptyset}) \quad = \quad \begin{pmatrix} 0 & \overrightarrow{E} \\ -\overrightarrow{E} & 0 \end{pmatrix} = \overrightarrow{E}\,, \tag{60}$$

where $\overrightarrow{E}$ and $\overrightarrow{\mathcal{H}}$ are the electric and magnetic fields given by Eqs. (51) and (52). Here we have the different split octonion representation in comparison to the case quaternion electrodynamics, where $\frac{1}{2}(\overline{D}\emptyset + D\overline{\emptyset})$ represents Lorentz condition (i.e. scalar part of quaternion) and $\frac{1}{2}\mathrm{i}\,(\overline{D}\emptyset - D\overline{\emptyset})$ represents the vector part (pure quaternion) of a quaternion. It is obvious because the off-diagonal elements of an octonion is a quaternion while those along principal diagonal are the scalars.

(ii) Tachyonic case

For the description of tachyons, we substitute $A_4 = B_4$, $\overrightarrow{A} = \overrightarrow{B}$ in Eqs. (40) to (49) and get the representation of T^4-space. In this space (one space and three time dimensions), the scalar component is $r = |\overrightarrow{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and the time gradient vector $\overrightarrow{\nabla}' = [t_x, t_y, t_z, \partial_4' = (-\mathrm{i}r)]$. Hence the split octonion form of four-differential operator, its conjugate and the four-potential of tachyons may be written as

$$D \quad = \quad \begin{pmatrix} \partial_4' & -\overrightarrow{\nabla}' \\ \overrightarrow{\nabla}' & \partial_4' \end{pmatrix}, \tag{61}$$

$$\overline{D} \quad = \quad \begin{pmatrix} \partial_4' & \overrightarrow{\nabla}' \\ -\overrightarrow{\nabla}' & \partial_4' \end{pmatrix}, \tag{62}$$

$$\emptyset = \begin{pmatrix} B_4 & -\overrightarrow{B} \\ \overrightarrow{B} & B_4 \end{pmatrix}. \tag{63}$$

Thus we get

$$\overline{D}\emptyset = F' = \begin{pmatrix} \partial_4' B_4 + \overrightarrow{\nabla}.\overrightarrow{B} & -(\partial_4'\overrightarrow{B} - \overrightarrow{\nabla}' B_4 - \overrightarrow{\nabla}' \times \overrightarrow{B}) \\ \partial_4'\overrightarrow{B} - \overrightarrow{\nabla} B_4 - \overrightarrow{\nabla}' \times \overrightarrow{B} & \partial_4' B_4 + \overrightarrow{\nabla}'.\overrightarrow{B} \end{pmatrix}. \tag{64}$$

Now, using the following expressions for tachyonic [20, 21] electric and magnetic fields in T^4- space as

$$\overrightarrow{E_t} = -\frac{\partial \overrightarrow{B}}{\partial r} - \overrightarrow{\nabla}' \emptyset_g \,, \tag{65}$$

$$\overrightarrow{H_t} = \overrightarrow{\nabla}' \times \overrightarrow{B} \,, \tag{66}$$

and imposing Lorentz gauge condition in T^4-space as

$$\partial_4' B_4 + \overrightarrow{\nabla}'.\overrightarrow{B} = 0 \,,$$

we get

$$-\partial_4'\overrightarrow{B} + \overrightarrow{\nabla}' B_4 + \overrightarrow{\nabla}' \times \overrightarrow{B} = -\mathrm{i}\overrightarrow{E_t} + \overrightarrow{H_t} = -\mathrm{i}\overrightarrow{\psi_t}* \,, \tag{67}$$

$$\partial_4'\overrightarrow{B} - \overrightarrow{\nabla} B_4 - \overrightarrow{\nabla}' \times \overrightarrow{B} = \mathrm{i}\overrightarrow{E_t} - \overrightarrow{H_t} = \mathrm{i}\overrightarrow{\psi_t}* \,, \tag{68}$$

where $\overrightarrow{\psi_t} = \overrightarrow{E_t} - \mathrm{i}\overrightarrow{H_t}$ (t denotes the tachyonic representations). Then Eq. (64) reduces to

$$F' = \begin{pmatrix} 0 & -\overrightarrow{F_t} \\ \overrightarrow{F_t} & 0 \end{pmatrix} = \mathrm{i}\overrightarrow{\psi_t} * . \tag{69}$$

Equation (64) is the split octonion form of field tensor $F'_{\mu\nu} = \partial'_\mu B'_\nu - \partial'_\nu B'_\mu$ components of which describe the electric and magnetic field in T^4- space. Similarly, we obtain

$$D\overline{\emptyset} = F_t' \,, \tag{70}$$

where

$$F_t' = \begin{pmatrix} 0 & \overrightarrow{F_t}* \\ -\overrightarrow{F_t}* & 0 \end{pmatrix} = \mathrm{i}\overrightarrow{\psi_t} . \tag{71}$$

In the same manner, we get

$$\frac{1}{2}(\overline{D}\emptyset + D\overline{\emptyset}) = \begin{pmatrix} 0 & \overrightarrow{H_t} \\ -\overrightarrow{H_t} & 0 \end{pmatrix} = \overrightarrow{H_t}\,, \tag{72}$$

$$\frac{1}{2}\mathrm{i}\,(\overline{D}\emptyset - D\overline{\emptyset}) = \begin{pmatrix} 0 & \overrightarrow{E_t} \\ -\overrightarrow{E_t} & 0 \end{pmatrix} = \overrightarrow{E_t}\,, \tag{73}$$

where $\overrightarrow{E_t}$ and $\overrightarrow{H_t}$ are the electric and magnetic fields of tachyons in T^4-space as given by Eqs. (65) and (66).

(iii) Dyonic case

In order to reformulate dyonic field equations in terms of split octonions and its Zorn's vector matrix realization, we replace $\emptyset$ in Eq. (42) by the complex (generalized) four-potential $\{V_\mu\}$ of dyons [28] in the following form

$$V = V_4 u_0 * + V_4 u_0 + V_i u_i * + V_i u_i = \begin{pmatrix} V_4 & -\overrightarrow{V} \\ \overrightarrow{V} & V_4 \end{pmatrix}, \tag{74}$$

where V_4 and $\overrightarrow{V}$ are the temporal and spatial components of generalized four-potential of dyons. These are complex quantities and their real and imaginary components are electric and magnetic constituents [28]. Split octonion four-differential operator is now defined as

$$D = \partial_4 u_0 * + \partial_4 u_0 + \partial_i u_i * + \partial_i u_i = \begin{pmatrix} \partial_4 & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & \partial_4 \end{pmatrix}. \tag{75}$$

Operating split octonion conjugate of four-differential operator given by Eq. (75) on the equation (74) and using to communication relations of split octonion units, we get

$$\overline{D}V = \begin{pmatrix} \partial_4 V_4 + \overrightarrow{\nabla}.\overrightarrow{V} & -(\partial_4 \overrightarrow{V} - \overrightarrow{\nabla} V_4 - \overrightarrow{\nabla} \times \overrightarrow{V}) \\ \partial_4 \overrightarrow{V} - \overrightarrow{\nabla} V_4 - \overrightarrow{\nabla} \times \overrightarrow{V} & \partial_4 V_4 + \overrightarrow{\nabla}.\overrightarrow{V} \end{pmatrix}, \tag{76}$$

where

$$\partial_4 V_4 + \overrightarrow{\nabla}.\overrightarrow{V} = (\partial_\mu V_\mu)(u_0 * + u_0) = 0\,, \tag{77}$$

and

$$-\partial_4 \overrightarrow{V} + \overrightarrow{\nabla} V_4 + \overrightarrow{\nabla} \times \overrightarrow{V} = -\mathrm{i}\psi *\,. \tag{78}$$

Thus V is the generalized split octonion potential of dyons and $\overrightarrow{\psi}$ is the complex vector field with $\overrightarrow{E}$ and $\overrightarrow{H}$ as the generalized electromagnetic fields of dyons. Consequently, Eq. (76) leads to the following split octonion wave equation for dyonic field as

$$\overline{D}V \;=\; G\,, \tag{79}$$

where

$$G \;=\; \begin{pmatrix} 0 & -\overrightarrow{G} \\ \overrightarrow{G} & 0 \end{pmatrix} = \mathrm{i}\overrightarrow{\psi} * . \tag{80}$$

It is to be noted from Eqs. (46) to (49) that f_4, $\overrightarrow{f}$, $\overrightarrow{f}\,'$ and f_4' are the components of electric and magnetic fields described in terms of two four-potentials in internal and external spaces. If the two spaces are completely disjoint spaces, then we get only $\overrightarrow{f}$ in external four-space since $f_4 = 0$ due to Lorentz gauge condition, while $\overrightarrow{f}\,'$ and f_4' do not occur. Equations (48) and (49) show that $\overrightarrow{f}$ and $\overrightarrow{f}'$ are made up from electric and magnetic fields, which is the mixing of external and internal spaces. As such, Eqs. (48) and (49) do not describe either the usual electric and magnetic fields or the generalized fields of dyons. Rather, they have the mixed behaviour, namely tachyonic dyons.

4.2. Field equations

Let us write the octonion wave equation in split representation as

$$DF \;=\; J\,. \tag{81}$$

Here D and F are defined by equations (40) and (45) in split octonionic form and J is associated with eight-dimensional current source density in split octonion representation as follows,

$$J \;=\; \begin{pmatrix} J_4 & -\overrightarrow{J} \\ \overrightarrow{J} & J_4 \end{pmatrix}. \tag{82}$$

Now, using Eqs. (40) and (45), we get

$$DF \;=\; \begin{pmatrix} \partial_4 f_4 - \overrightarrow{\nabla}.\overrightarrow{f}\,' & -(\partial_4 \overrightarrow{f} + \overrightarrow{\nabla} f_4' + \overrightarrow{\nabla}' \times f\,') \\ \partial_4' \overrightarrow{f}\,' + \overrightarrow{\nabla}' f_4 + \overrightarrow{\nabla} \times \overrightarrow{f} & \partial_4' f_4' - \overrightarrow{\nabla}'.\overrightarrow{f} \end{pmatrix}. \tag{83}$$

Comparing Eqs. (81), (82) and (83), we get

$$J_4 \;=\; \partial_4 f_4 - \overrightarrow{\nabla}.\overrightarrow{f}\,', \tag{84}$$

$$J'_4 \quad = \quad \partial'_4 f'_4 - \overrightarrow{\nabla}'.\overrightarrow{f}\,, \tag{85}$$

$$\overrightarrow{J} \quad = \quad \partial_4 \overrightarrow{f} + \overrightarrow{\nabla} f'_4 + \overrightarrow{\nabla}' \times f'\,, \tag{86}$$

$$\overrightarrow{J}' \quad = \quad \partial'_4 \overrightarrow{f}' + \overrightarrow{\nabla}' f_4 + \overrightarrow{\nabla} \times \overrightarrow{f}\,. \tag{87}$$

Let us discuss the following different cases of generalized field equation.

(i) Bradyonic case

For the description of bradyons, let us substitute $B_4 = A_4$, $\overrightarrow{B} = \overrightarrow{A}$, $\partial'_4 = \partial_4$, $\overrightarrow{\nabla}' = \overrightarrow{\nabla} \equiv \partial$ and $J'_4 = J_4$, $\overrightarrow{J}' = \overrightarrow{J}$. Using Eqs. (51), (52) and (56), we find that Eq. (83) reduces to

$$DF = \begin{pmatrix} \overrightarrow{\nabla}.\overrightarrow{H} + \mathrm{i}\overrightarrow{\nabla}.\overrightarrow{E} & \mathrm{i}\frac{\partial\overrightarrow{\mathcal{H}}}{\partial t} + \mathrm{i}\overrightarrow{\nabla}\times\overrightarrow{E} - \frac{\partial\overrightarrow{E}}{\partial t} + \overrightarrow{\nabla}\times\overrightarrow{\mathcal{H}} \\ -\mathrm{i}\frac{\partial\overrightarrow{\mathcal{H}}}{\partial t} - \mathrm{i}\overrightarrow{\nabla}\times\overrightarrow{E} + \frac{\partial\overrightarrow{E}}{\partial t} - \overrightarrow{\nabla}\times\overrightarrow{\mathcal{H}} & \overrightarrow{\nabla}.\overrightarrow{\mathcal{H}} + \mathrm{i}\overrightarrow{\nabla}.\overrightarrow{E} \end{pmatrix}. \tag{88}$$

Comparing it with Eqs. (81) and (82), we find that Eq. (88) is analogous to the split octonionic form of Maxwell's equation $F_{\mu\nu,\nu} = J_\mu$ which is the covariant form of a set of the following four-differential equations, i.e.

$$\overrightarrow{\nabla}.\overrightarrow{E} = \rho_e = -\mathrm{i}\,J_4, \quad \overrightarrow{\nabla}.\overrightarrow{H} = \quad 0\,,$$

$$\overrightarrow{\nabla} \times \overrightarrow{H} = -\overrightarrow{J} + \frac{\partial\overrightarrow{E}}{\partial t}, \quad \overrightarrow{\nabla} \times \overrightarrow{E} = \quad -\frac{\partial\overrightarrow{H}}{\partial t}\,. \tag{89}$$

Hence, the octonion field Eq. (81) is identical to the Maxwell's field equation in compact and simple split octonion form. Equation (81) may also be written in terms of potential as

$$DF \quad = D(\overline{D}\emptyset) \quad = D\overline{D}\emptyset = \Box\emptyset = J\,, \tag{90}$$

which is equivalent to the split octonion form of covariant field equation $\Box V_\mu = J_\mu$ of classical electrodynamics. As such, we have reformulated the classical electrodynamics in terms of compact, simple and consistent representation of split octonion formulations for the case of particles travelling slower than light, namely bradyons.

(ii) Tachyonic case

In this case, if we may substitute $A_4 = B_4$, $\overrightarrow{A} = \overrightarrow{B}$, $\partial_4 = \partial_4'$, $\overrightarrow{\nabla}' = \overrightarrow{\nabla} \equiv \partial$ and $J_4 = J_4'$, $\overrightarrow{J} = \overrightarrow{J}\,'$. Hence, using Eqs. (61) and (64), we get

$$DF \quad ' = \begin{pmatrix} J_4' & -\overrightarrow{J}\,' \\ \overrightarrow{J} & J_4 \end{pmatrix} = J\,'. \tag{91}$$

This equation is the split octonionic form of Maxwell's equation $F''_{\mu\nu,\nu} = J'_\mu$ in T^4-space where we have used the following pairs of Maxwell's equation for tachyons [24, 25]

$$\overrightarrow{\nabla}'.\overrightarrow{E}_t = -\rho_0, \quad \overrightarrow{\nabla}'.\overrightarrow{H_t} = \quad 0\,,$$

$$\overrightarrow{\nabla}' \times \overrightarrow{H_t} = -\overrightarrow{J}\,' + \frac{\partial \overrightarrow{E_t}}{\partial t}, \quad \overrightarrow{\nabla}' \times \overrightarrow{E_t} = \quad -\frac{\partial \overrightarrow{H_t}}{\partial t}\,. \tag{92}$$

Hence, Eq. (90) is visualized as the field equation of tachyons in T^4-space in compact, simple and consistent split octonionic formulation. It has already been concluded [20, 21] that T^4-space for tachyons plays the same role as bradyons do in R^4-space.

(iii) Dyonic case

Here we consider $\{A_\mu\}$ and $\{B_\mu\}$ as electric and magnetic four-potentials described in external space (i.e. R^4-space) only, D is defined by Eq. (75) and $\emptyset$ is replaced by V the generalized four- potential of dyons given by Eq. (74). Subsequently, on using Eqs. (75), (76) and (79), we get

$$D(\overline{D}V) \quad = \quad DG = J\,, \tag{93}$$

where

$$J \quad = J_4 u_0 * + J_4 u_0 + J_i u_i * + J_i u_i = \begin{pmatrix} J_4 & -\overrightarrow{J} \\ \overrightarrow{J} & J_4 \end{pmatrix} \tag{94}$$

is the split octonion form of generalized four-current associated with dyons. Accordingly, we get

$$D(\overline{D}V) \quad = (D\overline{D})V = DG \quad = J\,, \tag{95}$$

or equivalently

$$\Box V \quad = \quad J\,, \tag{96}$$

where

$$\Box \;=\; (\overline{D}D) = (D\overline{D}) = \partial_4^2 + \nabla^2 = \partial_i\partial_i \,. \tag{97}$$

Equations (93) and (95) are the split octonion equivalents of generalized Dirac-Maxwell's (GDM) equation of dyons. Hence, we may describe split octonion wave equation as the generalized field equation of dyons in compact, simple and consistent manner. Moreover, Eqs. (93) and (95) reproduce the field equations corresponding to the dynamics of electric charge (magnetic monopole) in the absence of magnetic (electric) charge on the dyons. Equation (95) is thus the split octonion equivalent of covariant field equations $\Box V_\mu = J_\mu$ of generalized fields of dyons described earlier [22].

4.3. *Current equation*

Let us discuss the octonion wave equation as the octonion current differential equations in eight-dimensions as

$$DJ \;=\; S \,. \tag{98}$$

Here D and J are defined earlier in Eqs. (41) and (82) in their split form, and $S = \Box F$ is defined as

$$S \;=\; \begin{pmatrix} S_4 & -\overrightarrow{S} \\ \overrightarrow{S}' & S_4' \end{pmatrix}. \tag{99}$$

Now using Eqs. (41) and (46) and equation (82) , we get

$$DJ \;=\; \begin{pmatrix} \partial_4 J_4 - \overrightarrow{\nabla}.\overrightarrow{J}\,' & -(\partial_4 \overrightarrow{J} + \overrightarrow{\nabla} J_4' + \overrightarrow{\nabla}' \times \overrightarrow{J}\,') \\ \partial_4' \overrightarrow{J}\,' + \overrightarrow{\nabla}' J_4 + \overrightarrow{\nabla} \times \overrightarrow{J} & \partial_4' J_4' - \overrightarrow{\nabla}'.\overrightarrow{J} \end{pmatrix}. \tag{100}$$

Now, using equations (98), (99) and (100) , we get the following set of equations

$$S_4 \;=\; \partial_4 J_4 - \overrightarrow{\nabla}.\overrightarrow{J}\,' , \tag{101}$$

$$S_4' \;=\; \partial_4' J_4' - \overrightarrow{\nabla}'.\overrightarrow{J} , \tag{102}$$

$$\overrightarrow{S} \;=\; \partial_4 \overrightarrow{J} + \overrightarrow{\nabla} J_4' + \overrightarrow{\nabla}' \times \overrightarrow{J}\,' , \tag{103}$$

$$\overrightarrow{S}' \;=\; \partial_4' \overrightarrow{J}\,' + \overrightarrow{\nabla}' J_4 + \overrightarrow{\nabla} \times \overrightarrow{J} , \tag{104}$$

where S is the new octonion variable parameter obtained from the operation of differential operator to current. Hence we may obtain a new kind of field equation in terms of the new parameter S and, accordingly, we may analyze the different cases for bradyons, tachyons and dyons.

Acknowledgements

The work is supported by Uttarakhand Council of Science and Technology, Dehradun. One of us (OPSN) is thankful to the Chinese Academy of Sciences and Third World Academy of Sciences for awarding him CAS-TWAS visiting scholar fellowship to pursue a research program in China. He is also grateful to Professor Tianjun Li for his hospitality at the Institute of Theoretical Physics, Beijing, China.

References

1) P. Jordan, J. von Neuman and E. P. Wigner, Ann. Math. **35** (1934) 29.

[2] A. Cayley, Phil. Mag. **26** (1845) 208; R. P. Graves, *Life of Sir William Rowan Hamilton*, 3 volumes, Arno Press, New York (1975); J. C. Baez, Bull. Amer. Math. Soc. **39** (2001) 145; J. H. Conway, *On Quaternions and Octonions: Their Geometry, Arithmetic,and Symmetry*, A. K. Peters Ltd, Massachusetts (2003).

[3] L. E. Dickson, Ann. Math. **20** (1919) 155; W. R. Hamilton, *Elements of quaternions*, Chelsea Publications Co., NY (1969); P. G. Tait, *An Elementary Treatise on Quaternions*, Oxford Univ. Press (1875).

[4] A. Pais, Phys. Rev. Lett. **7** (1961) 291.

[5] R. Penny, Am. J. Phys. **36** (1968) 871; Nuovo Cimento B **3** (1971) 95.

[6] K. Imaeda, H. Tachibana, M. Imaeda and S.Ohta, Nuovo Cimento B **100** (1987) 53; B **105** (1990) 1203; Bull. Okayama Univ. Sci. A **24** (1989) 181; K. Imaeda and H. Tachibana, Nuovo Cimento B **104** (1989) 91; Bull. Okayama Univ. Sci. A **19** (1984) 93; H. Tachibana, Ph. D. Thesis, Okayama Univ. Science (1990).

[7] G. C. Joshi, Lett. Nuovo Cimento **44** (1985) 449; A. J. Davis and G. C. Joshi, J. Math. Phys. **27** (1986) 3036; R. Foot and G. C. Joshi, Int. J. Theor. Phys. **28** (1989) 1449; Phys. Lett. B **199** (1987) 203.

[8] A. Gamba, J. Math. Phys. **8** (1967) 775; Nuovo Cimento A **111** (1998) 293; M. Gogberashvili, J. Phys. A: Math. Gen. **39** (2006) 7099.

[9] G. M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, Kluwer (1994).

[10] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Montreal Memorial Lecture Series in Mathematical Physics, vol. 2, Cambridge University Press (1995); C. H. Barton and A. Sudbery, *Magic Squares of Lie Algebras*, arXiv:math.RA/0001083.

[11] R. Dündarer, F. Gürsey and C. Tze, J. Math. Phys. **25** (1980) 1496; Nucl. Phys. B **266** (1986) 440; F. Gürsey and C Tze, *On the Role of Division, Jordan and Related Algebras in Particle Physics*, World Scientific, River Edge, NJ (1996); M Günaydin and F Gürsey, Lett. Nuovo Cimento **6** (1973) 407; M. Koca and N. Ozdes, J. Phys. A **22** (1989) 1469; S. Catto, *Exceptional Projective Geometries and Internal Symmetries*, hep-th/0302079.

[12] M. Günaydin and F. Gürsey, J. Math. Phys. **14** (1973) 1651; Phys. Rev. D **9** (1974) 3387.

[13] G. Domokos and S. Domokos-Kovesi, Phys. Rev. D **19** (1979) 2984.

[14] K. Morita, Progr. Theor. Phys. **65** (1981) 787; **73** (1985) 999.

[15] B. A. Bernevig, J. Hu, N. Toumbas and S. Cheng Zhang, *The Eight Dimensional Quantum Hall Effect and the Octonions*, arXiv:cond-mat/0306045.

[16] C. A. Manogue and A. Sudbery, Phys. Rev. D **40** (1989) 4073.

[17] T. Kugo and P. Townsend, Nucl. Phys. B **221** (1983) 357; J. M. Evans, Nucl. Phys. B **298** (1988) 92; J. Lukierski and F. Toppan, Phys. Lett. B **584** (2004) 315; **539** (2002) 266.

[18] K. S. Abdel-Khalek Mostafa, *Ring Division Algebras, Self Duality and Supersymmetry*, Ph. D. thesis arXiv:hep-th/0002155; J. Schray, *Octonions and Supersymmetry*, Ph. D. thesis, Department of Physics, Oregon State University (1994).

[19] A. M. Buoncristiani, J. Math. Phys. **14** (1973) 849.

[20] S. Marques and C. G. Oliveira, J. Math. Phys. **26** (1985) 3131; **29** (1988) 2127; Phys. Rev. D **36** (1987) 1716; S. Marques-Bonham, J. Math. Phys. **31** (1990) 1478; **32** (1991) 1383.

[21] M. C. Pant, P. S. Bisht, O. P. S. Negi and B. S. Rajput, Can. J. Phys. **78** (2000) 303; H. C. Chandola and B. S. Rajput, J. Math. Phys. **26** (1985) 208.

[22] O. P. S. Negi et al., Phys. Lett. B **105** (1981) 281; O. P. S. Negi and B. S. Rajput, Phys. Lett. B **113** (1982) 183; Lett. Nuovo Cimento **34** (1982) 180; Indian J. Pure and Appl. Phys. **2** (1982) 472.

[23] E. Recami and R. Mignani, Riv. Nuovo Cimento **4** (1974) 209; Lett. Nuovo Cimento **16** (1976) 449; G. Dattoli and R. Mignani, Lett. Nuovo Cimento **22** (1978) 65; V. Vysin, Lett. Nuovo Cimento **22** (1978) 76; E. Recami, Riv. Nuovo Cimento **9** (1986) 1-178; Found. Phys. **31** (2001) 1119; Physica A **252** (1998) 586.

[24] G. D. Maccarrone and E. Recami, Lett. Nuovo Cimento **34** (1982) 251; **28** (1980) 151; P. Caldirola, G. D. Maccarrone and E. Recami, Lett. Nuovo Cimento, **29** (1980) 241.

[25] H. C. Chandola and B. S. Rajput, Indian J. Pure and Appl. Phys. **22** (1984) 1; Lett. Nuovo Cimento **40** (1984) 277.

[26] P. A. M. Dirac, Proc. Roy. Soc. London A **133** (1931) 60; Phys. Rev. **74** (1948) 817; G. 't Hooft, Nucl. Phys. B **79** (1974) 276; A. M. Polyakov, JETP Lett. **20** (1974) 194.

[27] J. Schwinger, Science **165** (1978) 757; Phys. Rev. **144** (1986) 1087; **151** (1966) 1048; **151** (1966) 1055; **173** (1968) 1536; D. Zwanzinger, Phys. Rev. **176** (1968) 1489; Phys. Rev. D **3** (1971) 880; B. Julia and A. Zee, Phys. Rev. D **11** (1975) 2227; N. Craigie, *Theory and Detection of Magnetic Monopoles in Gauge Theories*, World Scientific, Singapore (1986).

[28] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Nuovo Cimento A **104** (1991) 337; Progr. Theor. Phys. **85** (1991) 157; Shalini Bisht, P. S. Bisht and O. P. S. Negi, Nuovo Cimento B **113** (1998) 1449; B. S. Rajput, S. R. Kumar and O. P. S.Negi, Lett. Nuovo Cimento **34** (1982) 180; **36** (1983) 75.

[29] V. Majernik, Physica **39** (2000) 9; V. V. Kravchenkov, *Applied Quaternion Analysis*, Helderman Verlag, Germany (2003).

[30] Z. K. Silagadze, *Feynman's Derivation of Maxwell's Equations and Extra Dimensions*, Ann. Foundation Louis de Broglie **27** (2002) 241; hep-th/0106235.

[31] P. S. Bisht and O. P. S. Negi, Indian Journal of Pure and Appl. Phys. **32** (1994) 193; **31** (1993) 292; Shalini Dangwal, P. S. Bisht and O. P. S. Negi, Int. J. Theor. Phys. **47** (2008) 2297; *Octonionic Gauge Formulation for Dyonic Fields*, hep-th/0608061.

TUMAČENJA OKTONIONSKE VALNE JEDNADŽBE

Raspravljamo tumačenja oktonionske valne jednadžbe u osmodimenzijskom prostoru-vremenu. Pokušavamo raspraviti oktonionsku jednadžbu polja kao jednadžbu za gibanje čestica koje istovremeno nose električan i magnetski naboj (tj. diona) u vanjskom i unutarnjem prostoru. Zaključujemo da komponenta oktonionskog potencijala nema svojstva kako poopćenog elektromagnetskog polja, tako i diona. Zapravo, ona ima miješana svojstva elektromagnetskih polja pridruženih vanjskom i unutarnjim prostorima. Također, smo pokušali istražiti podvojenu oktonionsku jednadžbu i njeno tumačenje u klasičnoj elektrodinamici, te smo izveli skladne i sažete oblike osmodimenzijskog potencijala i jednadžbe za struju za dione koje smo izveli preko Zornove vektorske matrice za podvojene oktonione. Predstavljajući vanjski četveroprostor kao prostor za smještaj tahiona, pokazujemo da se podvojena oktonionska jednadžba svodi na Maxwellovu jednadžbu (jednadžbu polja) za bradione u R^4-prostoru, a u odsustvu ovih za tahione u T^4-prostoru.