ISSN 1330—0016
CODEN FIZBE7

EXCHANGE ENERGY FOR HEAVY QUARK POTENTIAL

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Received 27 January 1992

Revised manuscript received 17 June 1992

UDC 539.12

Original scientific paper

In this paper we have derived an exact analytical expression for exchange energy of quarks corresponding to the potential $V(r, T) \sim -\frac{1}{r^n} e^{-2m_D(T)r}$ with $n = 1$ and 2 in the limit when chemical potential $\mu_\alpha < 0$ and $\mu_\alpha \gg T$. We have also computed numerically the same both in the non-relativistic and relativistic regions.

1. Introduction

One of the interesting problems in high temperature plasma phase of QCD is that of heavy quark potential. It is argued[1,2,3] that static colour charges are screened in high temperature in chiral symmetric phase of QCD. This argument is based on perturbative result which shows that the potential is of the form[1]

$$V(r, T) \sim -\frac{1}{r^n} e^{-2m_D(T)r}, \tag{1}$$

with $n = 2$ and m_D being Debye screening mass (one of the characteristic of hot QCD plasma). Further the analysis of the functional form of the potential in the

pure gauge sector[2-7] suggests that it would be appropriate to allow arbitrary value of integer n. For temperature close to critical temperature n approaches to 1 and for $T \gg T_c$ n approaches 2[1,2,4].

Numerical analysis also shows that slightly beyond the perturbative horizon the potential becomes screened[2,3]. One may recall that one of the salient features of QCD is the asymptotic freedom which means at short distance (or high momenta) the interaction is arbitraryly weak implying that the coupling may be expected to converge at high temperature and (or) high density. On this basis Quark-gluon plasma (QGP) may be treated as weakly interacting gas.

Now most important physical quantity in the phase transitional investigation of QGP is its energy density which consists of different parts viz., the free energy of quarks, gluons etc.

Over the last few years, the energy density of QGP with massless quarks[8,9] and massive quarks[10] has been calculated applying statistical thermodynamics.

The introduction of the heavy quark potential (cf. Eq. (1)) enables one to compute the exchange and correlation energy of quarks which should be added to the total energy density of QGP. The relevance and importance of calculation of exchange energy (although in different context) has been discussed in detail by Chin[11].

The purpose of this paper is to present exact analytical expression as well as numerical computation of the exchange energy of quarks corresponding to the potential (1) with both $n = 2$ and $n = 1$ in the non-relativistic limit. Also numerical results of exchange energy is obtained in the relativistic limit.

The organisation of this paper is as follows. In section 2, exact analytical formulae for exchange energy is obtained by applying second quantization method. Section 3 deals with numerical analysis and section 4 is kept for remarks.

2. *Derivation of exchange energy*

To find the formula of exchange energy corresponding to the potential (cf. (1)) we use the following standard thermodynamical relation[12]

$$\left(\frac{\partial \Omega}{\partial \lambda^2}\right)_{T,\mu,r} = \left\langle \frac{\partial H}{\partial \lambda^2} \right\rangle = \frac{1}{\lambda^2} \langle u \rangle \tag{2}$$

where λ is charge parameter, Ω represents thermodynamical potential, u describes interaction of quarks arising due to potential $\overline{V}$ and μ denotes chemical potential.

To describe the quarks in the state $\Psi_{p\sigma}$ we construct the following operators

$$\Psi = \sum_{p,\sigma} \Psi_{p\sigma}\, a_{p\sigma} \quad \text{and} \quad \Psi^{+} = \sum_{p,\sigma} \Psi^{*}_{p\sigma}\, a^{+}_{p\sigma} \tag{3}$$

$a_{p\sigma}$ and $a^+_{p\sigma}$ being annihilation and creation operators. The interaction operator U may now be written as

$$U = -\frac{1}{2}\int\int \Psi^+(r_1)\,\Psi^+(r_2)\,\frac{\lambda^2}{(r_1-r_2)^2}\,e^{-M_D r}\,\Psi(r_2)\,\Psi(r_1)\,dv_1\,dv_2 \qquad (4)$$

with $M_D = 2m_D$.

Use of equation (3) enables one to write equation (4) as

$$U = -\frac{1}{2}\sum \langle p_1' p_2'|\, U_{12}\,|p_1\, p_2\rangle\, a^+_{p_1'\sigma_1}\, a^+_{p_2'\sigma_2}\, a_{p_2\sigma_2}\, a_{p_1\sigma_1}. \qquad (5)$$

Summation in equation (5) is extended over all momenta and spin components. $\langle p_1' p_2'|\, U_{12}\,|p_1 p_2\rangle$ represents the matrix elements of the interaction

$$U_{12} = \frac{\lambda^2}{|r_1-r_2|^2}\,e^{-M_D(r_1-r_2)} \qquad (6)$$

between two quarks.

It is to be noted that the matrix elements can be calculated from purely orbital function (since the interaction is independent of spin)

$$\Psi_p = \frac{1}{V^{1/2}}\,e^{ip\cdot r/\hbar}. \qquad (7)$$

Again taking account of the fact that only those terms having two operators a, a^+ with same p and σ (i. e. $p_1' = p_1$, $p_2' = p_2$ and $\sigma_1 = \sigma_2 = \sigma$) will have non-zero matrix elements. We have

$$U = \frac{\lambda^2}{2V^2}\sum_{p_1\neq p_2}\sum_{\sigma} n_{p_1\sigma}\, n_{p_2\sigma}\int e^{i(p_1-p_2)\cdot(r_1-r_2)/\hbar}\,\frac{e^{-M_D(r_1-r_2)}}{|r_1-r_2|^2}\,dv_1 dv_2, \qquad (8)$$

where $n_{p\sigma} = a_{p\sigma}a^+_{p\sigma}$ is the occupation number. Now replacing summation over momenta by integration (as in finite volume quark momenta can have continuous values) one gets

$$U = \frac{\lambda^2 V}{2\,(2\pi\hbar)^6}\sum_{\sigma}\int\int\int \frac{n_{p_1\sigma}\, n_{p_2\sigma}}{r^2}\,e^{-M_D r}\,e^{i(p_1-p_2)\cdot r/\hbar}\,dv\,dp_1^3\,dp_2^3. \qquad (9)$$

Now writting the volume element dv in terms of polar coordinates and performing integration over Θ, φ we get after few steps of straightforward calculations

$$\int \frac{e^{-M_D r}\,e^{i(p_1-p_2)\cdot r/\hbar}}{r^2}\,dv = \frac{4\pi\hbar}{K}\tan^{-1}\frac{K}{\hbar M_D} \qquad (10)$$

with $K = p_1 - p_2$.

Again for ideal gas one can write (the particle in different states being independent)

$$\langle n_{p_1\sigma}\, n_{p_2\sigma}\rangle = \bar{n}_{p_1\sigma}\, \bar{n}_{p_2\sigma} \tag{11}$$

with

$$\bar{n}_{p\sigma} = [1 + e^{(\varepsilon-\mu_\alpha)\beta}]^{-1}, \tag{12}$$

where μ_α is the chemical potential of α component of quark.

Thus, use of equations (1), (10) and (11) enables one to write the thermodynamical potential Ω_{ex} corresponding to exchange energy as

$$\Omega_{ex} = \frac{\lambda^2 g_{qu} V}{(2\pi\hbar)^5} \int\int \frac{\bar{n}_{p_1}\, \bar{n}_{p_2}}{K} \tan^{-1} \frac{K}{\hbar M_D}\, d^3p_1\, d^3p_2, \tag{13}$$

where g_{qu} is the number of quark components and is equal to 12.

Now we consider the limiting case, $\mu_\alpha < 0$ and $|\mu_\alpha| \gg T$ and in this case from (12) we have

$$\bar{n}_{p_1}\, \bar{n}_{p_2} = e^{2\mu_\alpha\beta} e^{-(p_1^2+p_2^2)\beta/2m}. \tag{14}$$

Writting p_1, p_2 in terms of new variables we get the exchange part of thermodynamical potential density Ω_{ex}

$$\bar{\Omega}_{ex} = \frac{\lambda^2 g_{qu}}{(2\pi\hbar)^5} e^{2\mu_\alpha\beta}\, I \tag{15}$$

where

$$I = \int_0^\infty \int_0^\infty e^{-\frac{(4S^2+q^2)\beta}{4m}} q^{-1} \tan^{-1} \frac{q}{\hbar M_D}\, d^3q\, d^3S \tag{16}$$

with $q = p_1 - p_2$ and $S = \dfrac{p_1 + p_2}{2}$.

Evaluating the integrals in (16) we have[13]

$$I = \frac{4}{\pi^{3/2}} \left(\frac{m}{\beta}\right)^{5/2} e^{\beta\hbar^2 M_D^2/4m}\, \mathrm{erfc}\, (\beta/4m)^{1/2}\, \hbar M_D. \tag{17}$$

Finally $\bar{\Omega}_{ex}$ reads

$$\bar{\Omega}_{ex} = \frac{\lambda^2 g_{qu}}{8\hbar^5 (\pi)^{3/2}} (m/\beta)^{5/2}\, e^{\beta(2\mu_\alpha + \hbar^2 M_D^2/4m)}\, \mathrm{erfc}\, ((\beta/4m)^{1/2}\, \hbar M_D). \tag{18}$$

Again we have from standard thermodynamical relation, the exchange energy density

$$E_{ex} = -\mu_\alpha \frac{\partial\bar{\Omega}_{ex}}{\partial\mu_\alpha} + \beta \frac{\partial\bar{\Omega}_{ex}}{\partial\beta} + \bar{\Omega}_{ex}. \tag{19}$$

R. H. S. of (19) can be evaluated easily and is given by

$$E_{ex} = \frac{\lambda^2 g_{qu}}{8\hbar^5 \pi^{3/2}} \left(\frac{m}{\beta}\right)^{5/2} e^{2/3\mu\beta} \left[\left(-\frac{3}{2} + \frac{\hbar^2 M_D^3 \beta}{4m} e^{\frac{\hbar^2 M_D^2 \beta}{3m}}\right) \times \right.$$

$$\left. \times \operatorname{erfc}((\beta/4m)^{1/2} \hbar M_D) - \hbar M_D \left(\frac{\beta}{4m\pi}\right)^{1/2}\right], \tag{20}$$

where we have used the fact that each quark has baryon number 1/3.

As already mentioned[1,2], in the proximity of critical temperature, n is close to 1. The exchange energy density E'_{ex} with $n = 1$ in equation (1) can be calculated in a similar manner. However since the potential in this case ($n = 1$) obeys the following relationship with that of $n = 2$

$$\overline{V}' \sim \left(-\frac{e^{-M_D r}}{r}\right) = -\frac{dv}{dM_D}. \tag{21}$$

The exchange energy density with $n = 1$ may be obtained from the relation

$$\Omega'_{ex} = -\frac{d}{dM_D}(\overline{\Omega}_{ex}). \tag{22}$$

Thus one gets immediately the thermodynamical potential density Ω'_{ex} and hence the exchange energy density E_{ex} and are given by

$$\Omega'_{ex} = \frac{g_{qu} \lambda^2}{8\hbar^5 \pi^{3/2}} \left(\frac{m}{\beta}\right)^{5/2} e^{2/3\mu\beta} e^{\hbar^2 \frac{M_D^2 \beta}{2m}} \times$$

$$\times \left(\hbar (\beta/4\pi m)^{1/2} - \frac{\hbar^2 M_D \beta}{2m}\right) \operatorname{erfc}((\beta/4m)^{1/2} \hbar M_D) \tag{23}$$

and

$$E'_{ex} = \frac{g_{qu} \lambda^2}{8\hbar^5 \pi^{3/2}} \left(\frac{m}{\beta}\right)^{5/2} e^{2/3\mu\beta} \left[-\frac{\hbar^2 M_D^2 \beta}{4m\pi} + \left(\frac{\hbar^2 M_D \beta}{4m} - \hbar (\beta/4m\pi)^{1/2} + \right.\right.$$

$$\left.\left. + \frac{\hbar^3 M_D \beta}{2m} (\beta/4m)^{1/2}\right) e^{\frac{\hbar^2 M_D^2 \beta}{4m}} \operatorname{erfc}((\beta/4m)^{1/2} \hbar M_D)\right]. \tag{24}$$

3. *Numerical analysis*

To find the numerical values of exchange energy for different temperatures in the limiting case $\mu < 0$ and $|\mu_\alpha| \gg T$ we have used equations (20) and (24),

respectively, for $n = 2$ and $n = 1$. For computation we have taken $g_{qu} = 12$, $\hbar = 1$ and $\lambda^2 = 1$. Exchange energies have been computed for $M_D = 3T$ (reasons for choice of this value of M_D is explained in Ref. 1) and for each set we have taken quark mass $m = 100$ MeV and $|\mu| = 300$ MeV. Results presented in Tables 1 and 2 do not include the variation of m and (or) μ since it is observed that exchange energies in all cases change insignificantly for small variation of either m or μ.

TABLE 1.

E_{ex} (MeV)	T (MeV)	$E_{ex\,rel}$ (MeV)
-2.06×10^{-5}	20	
-8.254×10^{-1}	80	
−2.205	120	
−12.749	160	
−18.8107	180	
−26.2306	200	
−57.1915	240	
−85.3402	300	−32.318
−119.05	340	−76.215
−160.31	380	−159.42
−182.55	400	−222.50
	440	−409.03

Numerical values or E_{ex} and $(E_{ex})_{rel}$ for different temperature; $m = 100$ MeV, $|\mu| = 300$ MeV, $M_D = 3T$.

TABLE 2.

E'_{ex} (MeV)	T (MeV)	$E'_{ex\,rel}$ (MeV)
1.05×10^{-6}	20	
-4.80×10^{-1}	80	
−17.36	120	
−24.566	160	
−43.635	180	
−71.779	200	
−164.62	240	
−434.74	300	−286.42
−736.22	340	−595.87
−1165.00	380	1115.00
−1435.84	400	−1478.88
	440	−2470.38

Numerical values E'_{ex} and $(E'_{ex})_{rel}$ for different temperatures: $m = 100$ MeV, $|\mu| = 300$ MeV and $M_D = 3T$.

As already mentioned, the formulae (cf (20), (24)) are derived in the limit $\mu_\alpha \gg T$. It means the result is valid only in the nonrelativistic region. In the rela-

tivistic region ($|\mu_\alpha| \ll T$) the equation (20) (for $n = 2$) and the equation (24) (for $n = 1$) may be expressed as follows:

$$(E_{ex})_{rel} = \frac{6}{\pi^3} \int_0^\infty \int_0^\infty \frac{K_1^2 K_2^2}{(K_1 - K_2)} \tan^{-1} \frac{(K_1 - K_2)}{M_D} [1 + e^{(\varepsilon_1 - \mu_\alpha)\beta}]^{-1} \times$$

$$\times [1 + e^{(\varepsilon_2 - \mu_\alpha)\beta}]^{-1} \cdot [1 - \varepsilon_1 \beta [1 + e^{-(\varepsilon_1 - \mu_\alpha)\beta}]^{-1} -$$

$$- \varepsilon_2 \beta [1 + e^{-(\varepsilon_2 - \mu_\alpha)\beta}]^{-1}] \, dK_1 \, dK_2, \tag{25}$$

$$(E'_{ex})_{vdl} = \frac{6}{\pi^3} \int_0^\infty \int_0^\infty \frac{K_1^2 K_2^2}{[M_D^2 + (K_1 - K_2)^2]} [1 + e^{(\varepsilon_1 - \mu_\alpha)\beta}]^{-1}$$

$$\times [1 + e^{(\varepsilon_2 - \mu_\alpha)\beta}]^{-1} [1 - \varepsilon_1 \beta [1 + e^{-(\varepsilon_1 - \mu_\alpha)\beta}]^{-1} -$$

$$- \varepsilon_2 \beta [1 + e^{-(\varepsilon_2 - \mu_\alpha)\beta}]^{-1}] \, dK_1 \, dK_2. \tag{26}$$

$(E_{ex})_{rel}$ and $(E'_{ex})_{rel}$ have been evaluated with the help of Gaussian quadrature method for computation of multidimensional integrals.

The analysis shows exchange energies (both in relativistic and nonrelativistic cases) increase steadily with temperature.

4. *Discussion and conclusion*

In this note we have computed the exchange energy of quarks corresponding to the potential $V(r, T) \sim -\frac{1}{r^n} e^{-2 m_D(T) r}$ (for $n = 1$ and 2) both in the nonrelativistic and relativistic regions.

In view of the increased attention drawn by heavy quark potential, it would be interesting to find also the analytical expression of correlation energy and to investigate in detail whether these (exchange and correlation energies) have sensible consequences on the phase transitional aspects. Further quantitative understanding of the screening is very important[2,6,7] and to search whether the physical mechanism is responsible for colour confinement phase transition[6,7].

Acknowledgement

One of the authors (B. K. M.) is grateful to Dr. R. K. Roychoudhury of Indian Statistical Institute, Calcutta, for valuable suggestions and discussions.

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ENERGIJA IZMJENE ZA POTENCIJAL TEŠKOG KVARKA

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UDK 539.12

Originalni znanstveni rad

Dobiven je točan analitički izraz za energiju izmjene kvarkova koji odgovara potencijalu $V(r, T) \sim -\frac{1}{r^n} \exp(-2m_D(T)\, r)$ za $n = 1$ i $n = 2$ u limesu kada je kemijski potencijal $\mu_\alpha < 0$ i $\mu_\alpha \gg T$. Iste veličine izračunate su numerički i u nerelativističkom i relativističkom limesu.