ISSN 1330–0016
CODEN FIZBE7

EXACT SOLUTIONS OF SUPERSYMMETRIC NONLINEAR SCHRÖDINGER EQUATIONS AND COUPLED K-dV EQUATIONS

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Received 27 October 1993

UDC 530.145

Original scientific paper

In this communication we report certain types of exact solutions of supersymmetric nonlinear Schrödinger equations and coupled KdV-equations by making an ansatz for the solution in each case.

1. *Introduction*

During the last two decades, study of the nonlinear wave phenomena has made a remarkable stride (Scott et al. [1]). It has been confirmed that several nonlinear partial differential equations are widely applicable to the various nonlinear phenomena in physics. One must solve nonlinear equations to get a knowledge of the system but the methods of solving are very few up to this time. Each of the methods, viz., Inverse scattering method (Gardner et al. [2]), Hirota's method (Hirota [3]), Trace method (Wadati and Sawada [4]) and direct algebraic method (Hereman et al. [5]) has some constraints. Here we present certain type of exact solutions of supersymmetric nonlinear Schrödinger equation (NLSE, Kulish [6]) and of coupled K-dV equation (Hirota and Satsuma [7]) by making an ansatz for the solution in each case following the method suggested by Huibin and Kelin (Huibin and Kelin [8,9]).

2. Formulation

The supersymmetric NLSE's (Kulish [6]) read as:

$$iq_t = -q_{xx} + 2kq^+q^2 + k\Psi\Psi^+ - i\sqrt{k}\Psi\Psi_x \tag{1a}$$

$$i\Psi_t = -2\Psi_{xx} + kq^+q - i\sqrt{k}(2q\Psi^+_x + \Psi^+q_x) \tag{1b}$$

where $q(x,t)$ is the original field and $\Psi(x,t)$, $\Psi^+(x,t)$ are the fermionic counterparts introduced through supersymmetry. In the following we will be working with the real and imaginary parts of $(1a, b)$ and so we set

$$q = u_0 + iv_0 \tag{2a}$$

$$\Psi = u_1 + iv_1 \tag{2b}$$

whence we have the four nonlinear partial differential equations

$$u_{0t} = -v_{0xx} + k[2v_0(u_0^2 + v_0^2) + v_0(u_1^2 + v_1^2)] - \sqrt{k}[u_1u_{1x} - v_1v_{1x}] \tag{3a}$$

$$-v_{0t} = -u_{0xx} + k[2u_0(u_0^2 + v_0^2) + u_0(u_1^2 + v_1^2] + \sqrt{k}[v_1u_{1x} - u_1v_{1x}] \tag{3b}$$

$$-v_{1t} = -2u_{1xx} + ku_1(u_0^2 + v_0^2) + \sqrt{k}[2(u_0v_0 - u_0v_{1x}) + (u_1v_{0x} - v_1u_{0x})] \tag{3c}$$

$$u_{1t} = -2v_{1xx} + kv_1(u_0^2 + v_0^2) - \sqrt{k}[2(u_0u_{1x} + v_0v_{1x}) + (u_1u_{0x} + v_1v_{0x})]\,. \tag{3d}$$

We now look for the travelling wave solutions of (3a – d) that is, we assume that

$$u_0(x,t) = u_0(x - \lambda t) = u_0(\xi) \tag{4a}$$

$$v_0(x,t) = v_0(x - \lambda t) = v_0(\xi) \tag{4b}$$

$$u_1(x,t) = u_1(x - \lambda t) = u_1(\xi) \tag{4c}$$

$$v_1(x,t) = v_1(x - \lambda t) = v_1(\xi) \tag{4d}$$

where λ is velocity to be determined. Inserting (4) into (3), we get

$$-\lambda u_{0\xi} = -v_{0\xi\xi} + k[2v_0(u_0^2 + v_0^2) + v_0(u_1^2 + v_1^2)] - \sqrt{k}[u_1u_{1\xi} - v_1v_{1\xi}] \tag{5a}$$

$$\lambda v_{0\xi} = -u_{0\xi\xi} + k[2u_0(u_0^2 + v_0^2) + u_0(u_1^2 + v_1^2)] + \sqrt{k}[v_1u_{1\xi} + u_1v_{1\xi}] \tag{5b}$$

$$\lambda v_{1\xi} = -2u_{1\xi\xi} + ku_1(u_0^2 + v_0^2) + \sqrt{k}[2(v_0u_{1\xi} - u_0v_{1\xi}) + (u_1v_{0\xi} - v_1u_{0\xi})] \tag{5c}$$

$$-\lambda u_{1\xi} = -2v_{1\xi\xi} + kv_1(u_0^2 + v_0^2) - \sqrt{k}[2(u_0 u_{1\xi} + v_0 v_{1\xi}) + (u_1 u_{0\xi} + v_1 v_{0\xi})] \, . \quad (5d)$$

To the equations 5(a) – (d), following the method of Huibin and Kelin [8,9], we make the ansatzs

$$u_0 = \sum_{i=0}^{m} a_i (\tanh \mu)^i \, , \quad v_o = \sum_{i=0}^{m} b_i (\tanh \mu)^i \qquad (6a, b)$$

$$u_1 = \sum_{i=0}^{m} c_i (\tanh \mu)^i \, , \quad v_1 = \sum_{i=0}^{m} d_i (\tanh \mu)^i \qquad (6c, d)$$

where the integer m and parameters a_i, b_i, c_i, d_i $(i = 1, \ldots m)$ and μ are to be determined. The requirement that the highest power of the function $(\tanh \mu\xi)$ for the nonlinear term, say, $v_0 u_0^2$ (or $u_1 u_{1\xi}$) of 5(a) and that for the derivative term $v_{0\xi\xi}$ must be equal gives the following relation

$$\begin{array}{ll} m + 2 = 3m & [\text{or } 2m + 1 = 3m \\ \text{so here, } m = 1 & \text{so here } m = 1] \, . \end{array}$$

For the other equations of the set (5), we obtain $m = 1$. So the equations (6) can now be written as

$$u_0 = a \tanh(\mu\xi) \qquad (7a)$$

$$v_0 = b_1 + b_2 \tanh(\mu\xi) \qquad (7b)$$

$$u_1 = c \tanh(\mu\xi) \qquad (7c)$$

$$v_1 = d_1 + d_2 \tanh(\mu\xi) \qquad (7d)$$

where a, b_1, b_2, c, d_1, d_2 and μ are the parameters to be determined. Here in u_0 and u_1, we have dropped the parameters a_0 and c_0 and taken $a_1 = a$ and $c_1 = c$ in order to avoid complexities. In general, one can incorporate a_0, c_0. Inserting now equations (7) into (5) and equating the same power of $\tanh(\mu\xi)$, we get the following parametric equations

$$-\lambda a\mu = k[2b_1^3 + b_1 d_1^2] + \sqrt{k}[d_1 d_2]\mu \qquad (8a)$$

$$\lambda b_2 = \sqrt{k}(d, c) \qquad (8b)$$

$$\lambda c_2 = \sqrt{k}(2cb_1 - ad_1) \qquad (8c)$$

$$-\lambda c\mu = k(d_1 b_1^2) - \sqrt{k}(2b_1 d_2 + b_2 d_1)\mu \qquad (8d)$$

$$0 = 2b_2\mu^2 + k[4b_1^2 b_2 + 2b_2 b_1^2 + 2b_1 d_1 d_2 + b_2 d_1^2] - \sqrt{k}(c^2 - d_2^2)\mu \tag{8e}$$

$$0 = 2a\mu^2 + k[2ab_1^2 + ad_1^2] + \sqrt{k}(2ad_2 c)\mu \tag{8f}$$

$$0 = 4c\mu^2 + k(cb_1^2) + 3\sqrt{k}(cb_2 - ad_2)\mu \tag{8g}$$

$$0 = 4d_2\mu^2 + k(2b_1 b_2 d_1 + b_1^2 d_2) - 3\sqrt{k}(ac + b_2 d_2) \tag{8h}$$

$$\lambda a\mu = k[2b_1 b_2^2 + 2b_1 a^2 + 4b_1 b_2^2 + b_1 c^2 + b_1 d_2^2 + 2b_2 d_1 d_2] - \sqrt{k}(d_1 d_2)\mu \tag{8i}$$

$$-\lambda b_2\mu = k[4ab_1 b_2 + 2ad_1 d_2] - \sqrt{k}(d_1 c)\mu \tag{8j}$$

$$-\lambda d_2\mu = k(2b_1 b_2 c) + \sqrt{k}(ad_1 - 2cb_1) \tag{8k}$$

$$\lambda c\mu = k[a^2 d_1 + d_1 b_2^2 + 2b_1 b_2 d_2] + \sqrt{k}[2b_1 d_2 + b_2 d_1]\mu \tag{8l}$$

$$0 = -2b_2\mu^2 + k[2b_2 a^2 + 2b_2^3 + b_2(c^2 + d_2^2)] - \sqrt{k}[d_2^2 - c^2]\mu \tag{8m}$$

$$0 = -2a\mu^2 + k[2a^3 + 2ab_2^2 + a(c^2 + d_2^2)] - \sqrt{k}(2d_2 c)\mu \tag{8n}$$

$$0 = -4c\mu^2 + k[c(a^2 + b_2^2)] - 3\sqrt{k}[-b_2 c + ad_2]\mu \tag{8o}$$

$$0 = -4d_2\mu^2 + k[d_2(a^2 + b_2^2)] + 3\sqrt{k}[ac + b_2 d_2]\,. \tag{8p}$$

Since u_1, v_1 are fermionic, we must assume fermionic character for the coefficients c, d_1, d_2. Due to the fermionic character, it is important to note that $c^2 = d_1^2 = d_2^2 = 0$. Also note that u_0, v_0 are bosonic. Taking these into consideration, we obtain from (8)

$$\begin{aligned}
a &= \frac{\left[\frac{\lambda}{b_1} \pm \sqrt{68k}\right]\mu}{36k} \\
b_1 &= \pm\lambda/(2\sqrt{k}) \\
b_2 &= (\mu/\sqrt{k})\left[-\frac{1}{18} \pm (\lambda/36b_1)\sqrt{\frac{17}{k}}\right] \\
c &= \pm(\mu/9k)(A/B) \\
d_1 &= \pm 9\lambda B \\
d_2 &= \pm(\mu/9k)(A/B) \mp 9a(\sqrt{k}B\lambda) \\
\mu &= \pm(-\lambda^2/4)^{1/2}
\end{aligned}$$

and two constraint equations relating a, μ, λ, A, B and k

$$\begin{aligned}(\mu^2/81k^2)(A^2/B^2) &= \pm(\mu a\lambda/\sqrt{k}) \\ \text{and } \mu A^2 &= \mp(\mu A) \pm 81a\lambda(k^{3/2}B^2) \\ \text{where } A &= [(1/18) - (\lambda/36b_1)(17/k)^{1/2}]^{1/2} \\ B &= \left[\frac{2}{k}\left\{19/(18)^2 \mp (5\lambda/162b_1)(17/k)^{1/2} \mp (\lambda/36b_1)^2(17/k)\right\}\right]^{1/2} .\end{aligned}$$

We thus obtain one type of exact solutions of (1) with one arbitrary parameter μ or λ.

We next proceed to obtain exact solutions of the coupled K-dV equations suggested by Hirota and Satsuma [7] that describes the interactions of two long waves with different dispersions.

These equations look like

$$u_t - a(u_{xxx} + 6uu_x) = 2b\Phi\Phi_x \tag{9a}$$

$$\Phi_t + \Phi_{xxx} + 3u\Phi_x = 0 \tag{9b}$$

where a, b are arbitrary constants.

We now look for travelling wave solutions of (9) that is, we assume

$$u(x,t) = u(x - wt) = u(\xi) \tag{10a}$$

$$\Phi(x,t) = \Phi(x - wt) = \Phi(\xi) \tag{10b}$$

where w is velocity to be determined. Inserting (10) into (9), we get

$$-wu_\xi - a(u_{\xi\xi\xi} + 6uu_\xi) = 2b\Phi\Phi_\xi \tag{11a}$$

$$-w\Phi_\xi + \Phi_{\xi\xi\xi} + 3u\Phi_\xi = 0 \,. \tag{11b}$$

To the equations 11(a), (b) we again make the ansatz

$$u = \sum_{i=0}^{m} a_i(\tanh\mu\xi)^i \tag{12a}$$

$$\Phi = \sum_{i=0}^{m} b_i(tanh\mu\xi)^i \tag{12b}$$

where the integer m, a_i, b_i ($i = 1, \ldots m$) and μ are the parameters to be determined. The requirement that the highest power of the function $\tanh(\mu\xi)$ for the nonlinear term uu_ξ (or $\Phi\Phi_\xi$) of (11a) and that for the derivative term $u_{\xi\xi\xi}$ must be equal gives the following relation

$$2m + 1 = m + 3\,.$$

So here, $m = 2$. For equation 11(b) we also get $m = 2$. Hence the equations (12a), (12b) now take the form

$$u = a_0 + a_1 \tanh \mu\xi + a_2 \tanh^2 \mu\xi \qquad (13a)$$

$$\Phi = b_0 + b_1 \tanh \mu\xi + b_2 \tanh^2 \mu\xi \qquad (13b)$$

where a_0, b_0, a_1, b_1, a_2, b_2 and μ are the parameters to be determined. Inserting now (13) in (11) and equating the same power of $\tanh(\mu\xi)$, we get twelve parametric equations where we get inconsistency in solving the parameters. But if we retain the highest power of $\tanh(\mu)$ and the parameters a_1, b_1 then (13) look like

$$u = a_0 + a_2 \tanh^2 \mu\xi \qquad (14a)$$

$$\Phi = b_0 + b_2 \tanh^2 \mu\xi\,. \qquad (14b)$$

Inserting (14) in (11) and equating now the same power of $(\tanh \mu)$ we get following six parametric equations

$$-2wa_2 + 16aa_2\mu^2 - 12aa_0a_2 = 4bb_0b_2 \qquad (15a)$$

$$-2wb_2 - 16b_2\mu + 6a_0b_2 = 0 \qquad (15b)$$

$$-2a_2w - 40aa_2\mu^2 - 12aa_2^2 + 12aa_0a_2 = 4b(b_2^2 - b_0b_2) \qquad (15c)$$

$$2b_2 + 40b_2\mu^2 + 6a_2b_2 - 6a_0b_2 = 0 \qquad (15d)$$

$$24aa_2\mu^2 + 12aa_2^2 = -4bb_2^2 \qquad (15e)$$

$$24b_2\mu^2 + 6a_2b_2 = 0\,. \qquad (15f)$$

On solving, we get

$$\begin{aligned}
a_0 &= (1 + 8\mu^2)/3 \\
a_2 &= -4\mu^2 \\
b_0 &= \frac{1}{bb_2}[2\mu^2(2a - 1) - 16\mu^3 + 16\mu^4(1 + a)] \\
b_2 &= \pm\left[\frac{-24a\mu^4}{b}\right]^{1/2} \\
w &= (1 - 8\mu + 8\mu^2)\,.
\end{aligned}$$

Thus we obtain one type of exact solutions of (9) with one arbitrary parameter μ (or w) which are different from those obtained by Hirota and Satsuma [7].

3. Conclusion

In our above computations we have shown that the method suggested by Huibin and Kelin [8,9] is effective in obtaining exact solutions of non-linear partial differential equations. However, the question of stability of such solutions arises which is the matter of our present investigation and will be published elsewhere.

Acknowledgements

The author is indebted to Prof. A. N. Basu for constant encouragement and helpful discussions.

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TOČNA RJEŠENJA SUPERSIMETRIČNIH NELINEARNIH SCHRÖDINGEROVIH JEDNADŽBI I VEZANIH K-dV JEDNADŽBI

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Originalni znanstveni rad

U radu smo prikazali neke vrste točnih rješenja supersimetričnih nelinearnih Schrödingerovih jednadžbi i vezanih K-dV jednadžbi služeći se pretpostavkom o obliku rješenja u svakom pojedinom slučaju.