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BETHE-ANSATZ, YANG-BAXTER EQUATION AND FACTORIZATION OF S-MATRIX

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In this communication we remodel the analysis of the Bethe-ansatz for the nonlinear Schrödinger equation in an external field. We have constructed the n-particle wave function, demonstrated the factorization of the S-matrix for $n = 2$, 3 and 4 with the important difference that the basic 2-body S-matrix is not a single matrix but two matrices, and verified the Yang-Baxter equation.

1. *Introduction*

Quantization of nonlinear integrable system is a subject of utmost importance. The main approach to this problem is that of Korepin and Faddeev [1]. But a severe restriction of their methodology is that it is not applicable to non-ultralocal systems or to systems in an external field. The only other technique is that of Bethe-ansatz [2]. Roychowdhuri and Sen [3] made an attempt to solve the nonlinear Schrödinger equation in an external field following the idea of Bethe [4], Yang [5] and Wadati et al. [6]. We have extended the analysis to construct the n-particle state and have shown that the scattering matrix can be factorized with an important difference that the basic building block is not a single S-matrix but two different ones. The Yang-Baxter equation has also been verified.

2. Formulation

The equation under consideration is the nonlinear Schrödinger equation in the external field:

$$\mathrm{i}\Psi_t + \Psi_{xx} - 2c\left|\Psi\right|^2\Psi + eEX\Psi = 0. \tag{1}$$

The operator structure of the particular state can be expressed via the action of creation–annihilation operators on the vacuum. The corresponding wave function, for the n-particle system, satisfies the equations

$$-\sum_{i=1}^{n}\frac{\mathrm{d}^2 f_n}{\mathrm{d}X_i^2} + 2c\sum_i\sum_{j(j\neq i)}\delta\left(X_i - X_j\right)f_n - eE\left(\sum X_i\right)f_n = E_1 f_n \tag{2}$$

where E is the external electric field and E_1 is the eigenvalue. The boundary condition to be satisfied by f_n was deduced in Ref. 3, by integrating Eq. (1). For the simplest two particle system we set:

$$f_2(x_1 > x_2) \;=\; A_i(x_1+x_2)\left[\alpha_1^{12}e^{\mathrm{i}k(x_1-x_2)} + \alpha_2^{12}e^{-\mathrm{i}k(x_1-x_2)}\right] \tag{3}$$

$$f_2(x_1 < x_2) \;=\; A_i(x_1+x_2)\left[\alpha_1^{21}e^{-\mathrm{i}k(x_1-x_2)} + \alpha_2^{21}e^{\mathrm{i}k(x_1-x_2)}\right], \tag{4}$$

where A_i stands for the Airy function, the solution of stationary Schrödinger equation in an electric field.

Imposing the boundary condition and continuity of f_2 at $x_1 = x_2$, we obtain:

$$\begin{pmatrix}\alpha_1^{12}\\ \alpha_2^{12}\end{pmatrix} = \begin{pmatrix}\frac{c}{\mathrm{i}k} & 1+\frac{c}{\mathrm{i}k}\\ 1-\frac{c}{\mathrm{i}k} & -\frac{c}{\mathrm{i}k}\end{pmatrix}\begin{pmatrix}\alpha_1^{21}\\ \alpha_2^{21}\end{pmatrix} \tag{5}$$

or

$$\alpha_i^{12} = S_{ij}^{(1)}\alpha_j^{21}, \tag{6}$$

where each of the α's represents amplitude of the plane wave attached with it and c is a constant that appears in the Eq. (1), whence $S_{ij}^{(1)}$ represents the two-body S-matrix. For the construction of many–particle states we refer to Ref. 3 where Eq. (2) has been solved for one, two and three particles, by separating the centre of mass and relative motion, and subsequently imposing the boundary condition. There we observe that an n-particle solution of Eq. (2) contains $(n-1)$ Airy functions and one plane wave. So we take the clue from this computation for three particle state and make the following ansatz:

$$
\begin{aligned}
f_3(x_1 > x_2 > x_3) \quad &= \quad A_i(x_1+x_2)A_i(x_3)\left[\alpha_1^{123}e^{\mathrm{i}k(x_1-x_2)} + \alpha_2^{123}e^{-\mathrm{i}k(x_1-x_2)}\right] \\
&+ \quad A_i(x_2+x_3)A_i(x_1)\left[\alpha_3^{123}e^{\mathrm{i}k(x_2-x_3)} + \alpha_4^{123}e^{-\mathrm{i}k(x_2-x_3)}\right] \\
&+ \quad A_i(x_1+x_3)A_i(x_2)\left[\alpha_5^{123}e^{\mathrm{i}k(x_1-x_3)} + \alpha_6^{123}e^{-\mathrm{i}k(x_1-x_3)}\right] \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
f_3(x_2 > x_1 > x_3) \quad &= \quad A_i(x_2+x_1)A_i(x_3)\left[\alpha_1^{213}e^{\mathrm{i}k(x_2-x_1)} + \alpha_2^{213}e^{-\mathrm{i}k(x_2-x_1)}\right] \\
&+ \quad A_i(x_1+x_3)A_i(x_2)\left[\alpha_3^{213}e^{\mathrm{i}k(x_1-x_3)} + \alpha_4^{213}e^{-\mathrm{i}k(x_1-x_3)}\right] \\
&+ \quad A_i(x_2+x_3)A_i(x_1)\left[\alpha_5^{213}e^{\mathrm{i}k(x_2-x_3)} + \alpha_6^{213}e^{-\mathrm{i}k(x_2-x_3)}\right], \qquad (8)
\end{aligned}
$$

where each of α's represents amplitude of the plane wave attached with it.

Imposing the boundary condition and continuity of f_3 at $x_1 = x_2$, we obtain:

$$\alpha_i^{123} = (M_1)_{ij}\,\alpha_j^{213}$$

where the matrix $(M_1)_{ij}$ is given by

$$
(M_1)_{ij} = \left(\begin{array}{cc|cccc}
\frac{c}{\mathrm{i}k} & 1+\frac{c}{\mathrm{i}k} & 0 & 0 & 0 & 0 \\
1-\frac{c}{\mathrm{i}k} & -\frac{c}{\mathrm{i}k} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -\frac{2c}{\mathrm{i}k} & 0 & 1-\frac{2c}{\mathrm{i}k} & 0 \\
0 & 0 & 0 & \frac{2c}{\mathrm{i}k} & 0 & 1+\frac{2c}{\mathrm{i}k} \\
0 & 0 & 1+\frac{2c}{\mathrm{i}k} & 0 & \frac{2c}{\mathrm{i}k} & 0 \\
0 & 0 & 0 & 1-\frac{2c}{\mathrm{i}k} & 0 & -\frac{2c}{\mathrm{i}k}
\end{array}\right). \qquad (9)
$$

Equation (9) immediately suggests factorization. It is interesting to note that $(M_1)_{ij}$ contains two distinct 2-body S-matrices: $S^{(1)}$ of Eq. (6) and the new one

$$
S^{(2)} = \begin{pmatrix} -\frac{2c}{\mathrm{i}k} & 1-\frac{2c}{\mathrm{i}k} \\ 1+\frac{2c}{\mathrm{i}k} & \frac{2c}{\mathrm{i}k} \end{pmatrix}. \qquad (10)
$$

Further, the connection between the regions $x_2 < x_1 < x_3$ and $x_2 < x_3 < x_1$ yields:

$$\alpha_i^{213} = (M_2)_{ij}\,\alpha_j^{213}$$

where the matrix $(M_2)_{ij}$ is given by

$$(M_2)_{ij} = \begin{pmatrix} -\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1-\frac{2c}{\mathrm{i}k} & 0 \\ 0 & \frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1+\frac{2c}{\mathrm{i}k} \\ 0 & 0 & \frac{c}{\mathrm{i}k} & 1+\frac{c}{\mathrm{i}k} & 0 & 0 \\ 0 & 0 & 1-\frac{c}{\mathrm{i}k} & -\frac{c}{\mathrm{i}k} & 0 & 0 \\ 1+\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & \frac{2c}{\mathrm{i}k} & 0 \\ 0 & 1-\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & -\frac{2c}{\mathrm{i}k} \end{pmatrix}. \quad (11)$$

Again the whole system is composed of $S^{(1)}$ and $S^{(2)}$. The same is valid in the case of M_3, N_1, N_2 and N_3 arising from the connections between the regions:

$$x_2 < x_3 < x_1 \text{ and } x_3 < x_2 < x_1,$$

$$x_1 < x_2 < x_3 \text{ and } x_1 < x_3 < x_2,$$

$$x_1 < x_3 < x_2 \text{ and } x_3 < x_1 < x_2,$$

$$x_3 < x_1 < x_2 \text{ and } x_3 < x_2 < x_1,$$

respectively. One can see that we can go to the state (321) from the configuration (123) via two routes:

$$(123) \underset{N_1}{\rightarrow} (132) \underset{N_2}{\rightarrow} (312) \underset{N_3}{\rightarrow} (321)$$

and

$$(123) \underset{M_1}{\rightarrow} (213) \underset{M_2}{\rightarrow} (231) \underset{M_3}{\rightarrow} (321)$$

whence we observe by actual multiplication

$$M_1 M_2 M_3 = N_1 N_2 N_3.$$

This is the Yang-Baxter equation.

We now proceed to construct the 4-particle state which shows a new phenomenon, that of no-scattering between a pair represented by an identity matrix. For example, we obtain

$$\alpha_i^{1234} = \Lambda_{ij} \alpha_j^{2134},$$

$$\Lambda_{ij} = \begin{pmatrix} S^{(1)} & P & Q \\ \overline{P} & T & \overline{P} \\ Q & P & I \end{pmatrix},$$

where P is a 2×8 block of zeros, $\overline{P}$ the transpose of P, Q is a 2×2 null matrix, I is the 2×2 unit matrix and T is a 8×8 one. The structure of T is given below:

$$T = \begin{pmatrix} \frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1+\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 \\ 0 & -\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1-\frac{2c}{\mathrm{i}k} & 0 & 0 \\ 0 & 0 & \frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1+\frac{2c}{\mathrm{i}k} & 0 \\ 0 & 0 & 0 & -\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & 1-\frac{2c}{\mathrm{i}k} \\ 1-\frac{c}{\mathrm{i}k} & 0 & 0 & 0 & -\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 \\ 0 & 1+\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & \frac{2c}{\mathrm{i}k} & 0 & 0 \\ 0 & 0 & 1-\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & -\frac{2c}{\mathrm{i}k} & 0 \\ 0 & 0 & 0 & 1+\frac{2c}{\mathrm{i}k} & 0 & 0 & 0 & \frac{2c}{\mathrm{i}k} \end{pmatrix}.$$

The block of the 2×2 unit matrix represents no scattering between the corresponding particle positions. One can again think of several permutations leading to several identical configurations starting from (1234), such as

$$\begin{array}{ccccccccc} (1234) & \to & (1243) & \to & (1423) & \to & (1432) & \to \\ & \to & (4132) & \to & (4312) & \to & (4321); & \end{array}$$

and

$$\begin{array}{ccccccccc} (1234) & \to & (2134) & \to & (2314) & \to & (2341) & \to \\ & \to & (3241) & \to & (3421) & \to & (4321). & \end{array}$$

Again, we can formulate an analogue of the Yang-Baxter-like identity.

Lastly, we can write the ansatz for the n-particle wave function which is

$$\begin{aligned} f_n &= \sum_p A_i(x_i + x_j) A_i(x_k) \ldots A_i(x_p) \times \\ &\times \left[\alpha^{ijk\ldots p}_{P(ijk\ldots p)} e^{+\mathrm{i}k(x_i - x_j)} + \beta^{ijk\ldots p}_{P(ijk\ldots p)} e^{-\mathrm{i}k(x_i - x_j)} \right] \end{aligned}$$

where P denotes the permutation over the symbols $ijk \ldots p$, and the number of Airy - functions occurring is one less than the number of particles in the state and no two x_i, x_j are equal. It is interesting to note that we have only one momenta even in the n-particle wave function, in contrast to the usual Bethe-states for pure nonlinear Schrödinger equation.

3. Conclusions

Almost all the features of the Bethe-ansatz are retained in our considerations of systems in the external field. In the light of the factorization of the S-matrix, it is really interesting to study the triangle-like equations in detail.

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BETHEOVA POSTAVKA, YANG-BAXTEROVA JEDNADŽBA I FAKTORIZACIJA S-MATRICE

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U članku se uvodi nov pristup analizi Betheove postavke za nelinearnu Schrödingerovu jednadžbu u vanjskom polju. Konstruirana je n-čestična valna funkcija, dokazana faktorizacija S-matrice za $n = 2$, 3 i 4, s važnom razlikom da je osnovna dvočestična S-matrica sastavljena od dvije matrice, i potvrđena je Yang-Baxterova jednadžba.