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COLLECTIVE–FIELD EXCITATIONS IN THE CALOGERO MODEL

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We consider the large-N Calogero model in the Hamiltonian collective–field approach based on the $1/N$ expansion. The Bogomol'nyi limit appears and the corresponding equation for the semiclassical configuration gives the correct ground-state energy. Using the method of the orthogonal polynomial, we find the excitation spectrum of density fluctuations around the semiclassical solution for any value of the statistical parameter λ. The wave functions of the excited states are explicitly constructed as a product of Hermite polynomials in terms of the collective modes. The two–point correlation function is calculated as a series expansion in $1/\rho$ for any intermediate statistics.

1. Introduction

Recently, much attention has been paid to the quantum N–body Calogero model in $1+1$ space-time [1]. The Calogero model belongs to a large family of one-dimensional quantum integrable models. It also exhibits fractional statistics. Namely, its ground-state wave function is of a Jastrow type and can be visualized as the Laughlin wave function [2,3]. More recently, applying the collective–field approach to the fractional quantum Hall droplet dynamics, it has been shown that the density correlation function of the edge state interpolates to the correlation function of the Calogero model [4]. The Calogero model is closely related to the matrix models and exhibits an exact equivalence with the collective-field formulation of the $d = 1$ string theory at the classical level [5]. The collective-field theory with the cubic Hamiltonian and other higher-order terms in the $1/N$ expansion was recently used in deriving a new interesting soliton solution in $1 + 1$ dimension [6]. This effective theory can again be recognized as the $\lambda = 1/2$ collective-field formulation of the Calogero model. It is, therefore, of considerable interest to study the general features of the Calogero model in a collective-field theoretical framework. To the lowest order, the collective-field theory of the Calogero model was shown to correctly reproduce the first relevant term in the ground-state energy [7,8].

In the present paper, we extend these investigations by including higher-order terms in the $1/N$ expansion. We also study the properties of the Calogero model, concentrating on the excitation spectrum of small fluctuations around the semiclassical configuration $\rho_0(x)$, the corresponding wave functions and the first quantum corrections (the one-loop contribution) to the ground-state energy.

2. *Collective–field description of the Calogero model*

Before studying of the excitation spectrum, let us briefly recall the essential features of the collective-field approach to the Calogero model [8]. The approach is described by the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}x \rho(x)(\partial_x \pi)^2 + \frac{1}{2}\int \mathrm{d}x \rho(x)\left(\frac{\lambda-1}{2}\frac{\partial_x \rho(x)}{\rho(x)} + \lambda \fint \mathrm{d}y \frac{\rho(y)}{x-y}\right)^2 +$$

$$+\frac{\omega^2}{8}\int \mathrm{d}x \mathrm{d}y \rho(x)(x-y)^2\rho(y) - \frac{\lambda-1}{4}\int \mathrm{d}x \partial_x^2 \delta(x-y)|_{y=x}$$

$$-\frac{\lambda}{2}\int \mathrm{d}x \rho(x)\partial_x \frac{P}{x-y}|_{y=x}, \tag{1}$$

where a dimensionless constant λ determines the strength of the Calogero pair coupling through the relation $\lambda(\lambda-1) = g$. ω is the strength of a harmonic confinement potential. The collective field $\rho(x)$ is the continuum limit of the dynamical quantity

$$\rho(x) = \sum_{i=1}^{N} \delta(x - x_i), \tag{2}$$

where x_i are the positions of N spinless bosonic particles. $\pi(x)$ is canonical conjugate of the field $\rho(x)$:

$$[\partial_x \pi(x), \rho(y)] = -i\partial_x \delta(x-y). \tag{3}$$

From the definition (2) follows that the collective field obeys the normalization condition

$$\int \mathrm{d}x \rho(x) = N. \tag{4}$$

The first term in the collective Hamiltonian, quadratic in the conjugate momentum π, represents the kinetic energy of the system. The second term, rewritten as a complete square, emerges as a quantum collective-field potential. The last two terms represent a singular contribution, but, as will be shown later, they are canceled by infinity zero-point fluctuations of the collective field ρ. To find the ground-state energy and the corresponding collective motion of the Calogero system in the large-N limit, we should minimize the energy functional (1) with respect to π and ρ, obeying the normalization condition (4). However, in our case, owing to the special features of the model, there is a much more efficient method of solving the problem.

Performing the $1/N$ expansion of the collective field $\rho(x)$ in the form

$$\rho(x) = \rho_0(x) + \eta(x), \tag{5}$$

where $\rho_0(x)$ is the ground-state semiclassical configuration and $\eta(x)$ a small density fluctuation around $\rho_0(x)$, we can rewrite the collective Hamiltonian (1) up to the quadratic terms in π and η as

$$\begin{aligned}
H = \frac{1}{2}\int \mathrm{d}x \rho_0(x) \left(\frac{\lambda-1}{2} \frac{\partial_x \rho_0(x)}{\rho_0(x)} + \lambda \fint \mathrm{d}y \frac{\rho_0(y)}{x-y} - \omega \sqrt{\frac{N}{2}} x \right)^2 \\
+ N\omega \sqrt{\frac{N}{2}} \left(\frac{\lambda N}{2} - \frac{\lambda-1}{2} \right) + \\
+ \frac{1}{2}\int \mathrm{d}x \rho_0(x) (\partial_x \pi)^2 + \frac{\pi^2 \lambda^2}{2} \int \mathrm{d}x \rho_0(x) \eta^2(x) \\
+ \frac{(\lambda-1)^2}{8} \int \mathrm{d}x \frac{(\partial_x \rho_0(x))^2}{\rho_0^3(x)} \eta^2(x) + \\
+ \frac{(\lambda-1)^2}{8} \int \mathrm{d}x \partial_x \left(\frac{\partial_x \rho_0(x)}{\rho_0^2(x)} \right) \eta^2(x) \\
+ \frac{(\lambda-1)^2}{8} \int \mathrm{d}x \frac{(\partial_x \eta(x))^2}{\rho_0(x)} + \frac{\lambda(\lambda-1)}{2} \fint \mathrm{d}x \mathrm{d}y \frac{\partial_x \eta(x) \eta(y)}{x-y} + \\
+ \frac{\omega^2}{8} \int \mathrm{d}x \mathrm{d}y \eta(x)(x-y)^2 \eta(y) - \frac{\lambda-1}{4} \int \mathrm{d}x \partial_x^2 \delta(x-y)|_{y=x}
\end{aligned}$$

$$-\frac{\lambda}{2}\int \mathrm{d}x\rho_0(x)\partial_x \frac{P}{x-y}|_{y=x}. \tag{6}$$

There are no terms in the Hamiltonian (6) linear in $\eta(x)$ as we expand around the minimum of the dominant, large-N collective potential.

Owing to the positive definiteness of the first leading term, the Bogomol'nyi limit appears. The Bogomol'nyi bound is saturated by the positive normalizable solution $\rho_0(x)$ of the equation

$$\frac{\lambda-1}{2}\frac{\partial_x\rho_0(x)}{\rho_0(x)} + \lambda\fint \mathrm{d}y\frac{\rho_0(y)}{x-y} = \omega\sqrt{\frac{N}{2}}x, \tag{7}$$

with the ground-state energy equal to

$$E_0 = \frac{\omega}{2}\sqrt{\frac{N}{2}}[\lambda N(N-1)+N]. \tag{8}$$

We have not been able to obtain an analytic solution to this equation for any value of λ. However, we can offer the asymptotic behaviour of the semiclassical configuration $\rho_0(x)$. For $\lambda\neq 1$, the expression for $\rho_0(x)$ near the origin is

$$\rho_0(x) = A\exp\left[\frac{x^2}{\lambda-1}(\omega\sqrt{\frac{N}{2}}+\lambda B)\right], \tag{9}$$

where A and B are positive arbitrary constants. For $x\to\infty$, we obtain

$$\rho_0(x) = Cx^{2\lambda N/(1-\lambda)}\exp\left(\omega\sqrt{\frac{N}{2}}\frac{x^2}{\lambda-1}\right), \tag{10}$$

where C is again a positive constant. It is evident that the character of the solution depends crucially on the value of the parameter λ. For $\lambda>1$, the solution ρ_0 exists on the compact support only. As is well known, the coupling constant λ specifies the statistics of the Calogero model. For special values of λ, i.e. $\lambda=0$, we have bosons and for $\lambda=1/2, 1$ and 2 the model is related to the systems of orthogonal, unitary (fermions) and symplectic matrix theories, respectively [8]. For the bosonic and fermionic case, Eq. (7) can be exactly solved. The normalized bosonic distribution, $\lambda=0$, is given by

$$\rho_0(x) = N\sqrt{\frac{\omega}{\pi}\sqrt{\frac{N}{2}}}\exp\left(-\omega\sqrt{\frac{N}{2}}x^2\right). \tag{11}$$

It has a tail, in contrast to the fermionic distribution, $\lambda=1$:

$$\rho_0(x) = \sqrt{\frac{\omega}{\pi}\sqrt{\frac{N}{2}}\left(2N-\omega\sqrt{\frac{N}{2}}x^2\right)}, \tag{12}$$

which has a sharp boundary and is defined on the compact support only.

2.1. The reduced Calogero model

Let us now find an interesting collective-field hole excitation of the Calogero model, which can also be reached by the Bogomol'nyi saturation. Using the identity for the principal distribution:

$$\frac{P}{x-y}\frac{P}{x-z}+\frac{P}{y-x}\frac{P}{y-z}+\frac{P}{z-x}\frac{P}{z-y}=\pi^2\delta(x-y)\delta(x-z), \tag{13}$$

and performing partial integration, we can rewrite the first, leading term in the Hamiltonian (6) as

$$\begin{aligned}&\frac{1}{2}\int \mathrm{d}x\rho_0(x)\left(\frac{\lambda-1}{2}\frac{\partial_x\rho_0(x)}{\rho_0(x)}+\lambda\,⨍\mathrm{d}y\frac{\rho_0(y)}{x-y}-\omega\sqrt{\frac{N}{2}}x\right)^2=\\ &=\frac{1}{2}\int \mathrm{d}x\rho_0(x)\left(\frac{\lambda-1}{2}\frac{\partial_x\rho_0(x)}{\rho_0(x)}+\lambda\,⨍\mathrm{d}y\frac{\rho_0(y)}{x-y}-\omega\sqrt{\frac{N}{2}}x+\frac{c}{x}\right)^2-\\ &-\left(\frac{c^2}{2}+\frac{c(\lambda-1)}{2}\right)\int\mathrm{d}x\frac{\rho_0(x)}{x^2}\\ &+\omega cN\sqrt{\frac{N}{2}}+\frac{c\lambda}{2}\left(\int\mathrm{d}x\frac{\rho_0(x)}{x}\right)^2-\frac{c\lambda\pi^2}{2}\rho_0^2(0).\end{aligned} \tag{14}$$

For the symmetric configuration, $\rho_0(x)=\rho_0(-x)$, representing a hole located at the origin, $\rho_0(0)=0$, and the particular value of the constant c given by

$$c=1-\lambda,$$

the Bogomol'nyi limit appears. The contribution of the squared term in (14) vanishes and the corresponding configuration satisfies the enlarged Bogomol'nyi equation

$$\frac{\lambda-1}{2}\frac{\partial_x\rho_0(x)}{\rho_0(x)}+\lambda\,⨍\mathrm{d}y\frac{\rho_0(y)}{x-y}-\omega\sqrt{\frac{N}{2}}x+\frac{1-\lambda}{x}=0. \tag{15}$$

The role of the new, singular term in Eq. (15) is to compensate for the singularity produced by $\partial_x\ln\rho_0(x)$ at the origin, $x=0$. The corresponding energy is given by

$$E=E_0+N\sqrt{\frac{N}{2}}\omega(1-\lambda), \tag{16}$$

and can be lower than E_0 for $\lambda>1$. A more detailed discusion of the Bogomol'nyi Eq. (15) will be given elsewhere. From now on we restrict our investigation to the case of the ground-state semiclassical configuration ρ_0 given by Eq. (7).

3. *Diagonalisation*

To find the spectrum of low-lying excitations, we have to diagonalise part of the collective Hamiltonian (6) that is quadratic in the operators π and η. Let us first eliminate $\rho_0(x)$ in the kinetic-energy part by introducing a modified fluctuation $\tilde{\eta}(x)$ through

$$\eta(x) = \partial_x \left(\sqrt{\rho_0(x)}\tilde{\eta}(x) \right). \tag{17}$$

In this case the kinetic energy transforms as

$$\frac{1}{2} \int \mathrm{d}x \rho(x) (\partial_x \pi)^2 \quad \longrightarrow \quad -\frac{1}{2} \int \mathrm{d}x \frac{\delta^2}{\delta \tilde{\eta}^2(x)}. \tag{18}$$

Next we introduce the normal mode expansion of the fluctuation $\tilde{\eta}$

$$\tilde{\eta}(x) = \frac{1}{\sqrt{\rho_0(x)}} \sum_n q_n \varphi_n(x), \tag{19}$$

and the corresponding conjugate momentum

$$\tilde{\pi}(x) = -i \frac{\delta}{\delta \tilde{\eta}(x)} = \frac{1}{\sqrt{\rho_0(x)}} \sum_n p_n \varphi_n(x), \tag{20}$$

where the operators q_n and p_n satisfy standard bosonic commutator algebra

$$[q_n, p_m] = i\delta_{nm}, \quad [q_n, q_m] = [p_n, p_m] = 0. \tag{21}$$

The yet unspecified functions $\varphi_n(x)$ form an orthonormal and complete set in the sense that the following relations are satisfied:

$$\int \mathrm{d}x \frac{\varphi_n(x)\varphi_m(x)}{\rho_0(x)} = \delta_{nm}, \tag{22a}$$

$$\sum_n \frac{\varphi_n(x)\varphi_n(y)}{\sqrt{\rho_0(x)\rho_0(y)}} = \delta(x-y). \tag{22b}$$

Substituting the relations (19) and (20) into the Hamiltonian (6) and using Eq. (22a), we obtain the Hamiltonian in the diagonal form:

$$H = E_0 + \frac{1}{2} \sum_n p_n^2 + \frac{1}{2} \sum_n \omega_n^2 q_n^2 -$$

$$-\frac{\lambda - 1}{4} \int \mathrm{d}x \partial_x^2 \delta(x-y)|_{y=x} - \frac{\lambda}{2} \int \mathrm{d}x \rho_0(x) \partial_x \frac{P}{x-y}|_{y=x}, \tag{23}$$

if the function $\varphi_n(x)$ satisfies the integro-differential equation

$$\frac{(\lambda-1)^2}{8}\rho_0(x)\partial_x^2\left(\frac{\partial_x^2\varphi_n(x)}{\rho_0(x)}\right)-\frac{\pi^2\lambda^2}{2}\rho_0(x)\partial_x(\rho_0(x)\partial_x\varphi_n(x))-$$

$$-\frac{(\lambda-1)^2}{8}\rho_0(x)\partial_x\left\{\left[\frac{(\partial_x\rho_0(x))^2}{\rho_0^3(x)}+\partial_x\left(\frac{\partial_x\rho_0(x)}{\rho_0^2(x)}\right)\right]\partial_x\varphi_n(x)\right\}+$$

$$+\frac{\lambda(\lambda-1)}{2}\rho_0(x)\partial_x^2 \fint \mathrm{d}y\frac{\partial_y\varphi_n(y)}{x-y}-\frac{\omega^2}{4}\rho_0(x)\int \mathrm{d}y\varphi_n(y)=\frac{\omega_n^2}{2}\varphi_n(x). \tag{24}$$

It remains to prove that the corresponding eigenfrequencies ω_n are given by non-negative numbers.

We can avoid the manipulation with this cumbersome equation if we realize that it can be rederived by the simpler eigenequation

$$\omega_n\varphi_n(x)=\rho_0(x)\lambda \fint \mathrm{d}y\frac{\partial_y\varphi_n(y)}{x-y}+\rho_0(x)\frac{\lambda-1}{2}\partial_x\left(\frac{\partial_x\varphi_n(x)}{\rho_0(x)}\right)-$$

$$-\frac{\omega}{\sqrt{2N}}\rho_0(x)\int \mathrm{d}y\varphi_n(y). \tag{25}$$

Substituting the expression (25) for φ_n into the right-hand side of the same expression and using the identity for the principal distributions (13), we easily obtain the eigenequation (24). The solutions to Eq. (25) indeed form an orthogonal set of functions in the interval $-\infty<x<\infty$. To prove the orthogonality (22a), we write the equation for $\varphi_n(x)$, multiply it by $\varphi_m(x)$, and then integrate over the interval

$$\omega_n\int\frac{\varphi_n(x)\varphi_m(x)}{\rho_0(x)}\mathrm{d}x=\lambda \fint \mathrm{d}y\mathrm{d}x\frac{\partial_y\varphi_n(y)\varphi_m(x)}{x-y}+$$

$$+\frac{\lambda-1}{2}\int \mathrm{d}x\varphi_m(x)\partial_x\left(\frac{\partial_x\varphi_n(x)}{\rho_0(x)}\right)-\frac{\omega}{\sqrt{2N}}\int \mathrm{d}x\varphi_n(x)\int \mathrm{d}y\varphi_m(y). \tag{26}$$

If we now write Eq. (26) with n and m interchanged, subtracting it from Eq. (26), and integrating by parts, we obtain the orthogonality condition

$$(\omega_n-\omega_m)\int \mathrm{d}x\frac{\varphi_n(x)\varphi_m(x)}{\rho_0(x)}=0. \tag{27}$$

For $n\neq m$, the integral must vanish. Technically, the proof of the completeness of the set $\{\varphi_n(x)\}$ in (22b) is outside the scope of the present paper. We, therefore, omit it and simply anticipate that the solutions to Eq. (25) really satisfy the closure relation (22b).

3.1. The correlation functions

We are now in a position to calculate the inverse of the density-density correlation function in the Calogero model. The vacuum state of the operator part of the Hamiltonian (23) is given by

$$|0\rangle = \exp\left(-\frac{1}{2}\sum_n \omega_n q_n^2\right). \tag{28}$$

This may be reexpressed in terms of $\tilde{\eta}(x)$ using an expression inverse to Eq. (19)

$$q_n = \int \mathrm{d}x\tilde{\eta}(x)\frac{\varphi_n(x)}{\sqrt{\rho_0(x)}}, \tag{29}$$

and further simplified using the eigenequation (25). We finally obtain

$$|0\rangle = \exp\left(-\frac{1}{4}\int \mathrm{d}x\mathrm{d}y\eta(x)G^{-1}(x,y)\eta(y)\right), \tag{30}$$

where the inverse of the correlation function is given by

$$G^{-1}(x,y) = -2\lambda \ln|x-y| - (\lambda-1)\frac{\delta(x-y)}{\rho_0(x)} - \omega\sqrt{\frac{2}{N}}xy. \tag{31}$$

By functional integration over η, we can easily check that $G^{-1}(x,y)$ indeed represents the inverse of the correlation function $G(x,y)$:

$$\begin{aligned}\langle 0|\rho(x)\rho(y)|0\rangle &= \rho_0(x)\rho_0(y) + \langle 0|\eta(x)\eta(y)|0\rangle = \\ &= \rho_0(x)\rho_0(y) + \frac{\int \mathcal{D}\eta\eta(x)\eta(y)\exp\left(-\frac{1}{2}\int \eta G^{-1}\eta\right)}{\int \mathcal{D}\eta\exp\left(-\frac{1}{2}\int \eta G^{-1}\eta\right)} = \\ &= \rho_0(x)\rho_0(y) + G(x,y),\end{aligned} \tag{32}$$

where we have used the fact that the vacuum expectation value of the fluctuation η vanishes.

Before proceeding, we should point out that $G^{-1}(x,y)$ is determined up to the symmetric combination $f(x) + f(y)$ where $f(x)$ is an arbitrary real function. This is because of the normalization condition (4), the expansion (5) and the symmetry structure of G and G^{-1} (31).

According to the expression for the correlation function $G(x,y)$, Eq. (32), we can rewrite the eigenequation (25) as

$$\omega_n\varphi_n(x) = \frac{1}{2}\rho_0(x)\int \mathrm{d}y\varphi_n(y)\partial_x\partial_y G^{-1}(x,y). \tag{33}$$

Multiplying it by $\varphi_n(z)$ and summing over n we obtain

$$G(x,y) = \frac{1}{2}\partial_x\partial_y \sum_n \frac{\varphi_n(x)\varphi_n(y)}{\omega_n}. \tag{34}$$

Equations (34) and (25) imply that the correlation function $G(x,y)$ satisfies the following integro-differential equation:

$$2\lambda \fint \frac{\mathrm{d}y G(y,z)}{x-y} + (\lambda-1)\partial_x\left(\frac{G(x,z)}{\rho_0(x)}\right) = \partial_z\delta(x-z). \tag{35}$$

We can convert this equation into the differential equation using the identity for the principal distribution (13),

$$G(x,y) = -\frac{\lambda-1}{4\lambda^2\pi^2}\partial_x\left[\frac{1}{\rho_0(x)}\partial_x\left((\lambda-1)\frac{G(x,y)}{\rho_0(x)} + \delta(x-y)\right)\right] +$$

$$\frac{1}{2\lambda\pi^2\rho_0(x)}\partial_y\left(\frac{\rho_0(y)}{y-x}\right). \tag{36}$$

By rescaling the collective field and the correlation function as

$$\rho_0(x) = \frac{\lambda-1}{\lambda}\tilde{\rho_0}(x), \tag{37a}$$

$$G(x,y) = \frac{1}{\lambda}\tilde{G}(x,y), \tag{37b}$$

and iterating Eq. (36) (with respect to $G(x,y)$), we can compute the correlation function in terms of $\rho_0(x)$ to arbitrary order:

$$\tilde{G}(x,y) = \frac{1}{2\pi^2\tilde{\rho_0}(x)}\partial_y\left(\frac{\tilde{\rho_0}(y)}{y-x}\right) - \frac{1}{4\pi^2}\partial_x\left(\frac{1}{\tilde{\rho_0}(x)}\partial_x\delta(x-y)\right)$$

$$-\frac{1}{8\pi^4}\partial_x\left\{\frac{1}{\tilde{\rho_0}(x)}\partial_x\left[\frac{1}{\tilde{\rho_0}^2(x)}\partial_y\left(\frac{\tilde{\rho_0}(y)}{y-x}\right)\right]\right\}$$

$$+\frac{1}{16\pi^4}\partial_x\left\{\frac{1}{\tilde{\rho_0}(x)}\partial_x\left[\frac{1}{\tilde{\rho_0}(x)}\partial_x\left(\frac{1}{\tilde{\rho_0}(x)}\partial_x\delta(x-y)\right)\right]\right\} + \cdots. \tag{38}$$

Equation (38) is basically the $1/\rho_0$ expansion and we anticipate that it converges. We stress that expression (38) holds for λ different from zero and one, i.e. for generic intermediate statistics.

At this point we would like to analyse the structure of the static correlation function $G(x,y)$ in the reduced Calogero model, i.e. in the model without confining harmonic interaction ($\omega = 0$). In this case, the solution to Eq. (7) is given by the constant-density configuration $\rho = \rho_0$. It can be readily shown that $G(x,y)$ satisfies the following second-order differential equation

$$\left(\partial_x^2 + \frac{4\lambda^2\pi^2\rho_0^2}{(\lambda-1)^2}\right) G(x,y) = -\frac{\rho_0}{\lambda-1}\partial_x^2\delta(x-y) + \frac{2\lambda\rho_0^2}{(\lambda-1)^2}\partial_x\left(\frac{1}{x-y}\right). \qquad (39)$$

Having in mind the translation-invariance of the correlation function, let us now rewrite Eq. (39) in momentum space by Fourier transforming the function $G(x-y)$:

$$G(x-y) = \frac{\rho_0}{2\pi}\int \mathrm{d}k e^{\mathrm{i}k(x-y)}\tilde{G}(k), \qquad (40)$$

and the principal-value distribution

$$\frac{P}{x-y} = \frac{1}{2\mathrm{i}}\int \mathrm{d}k e^{\mathrm{i}k(x-y)}\mathrm{sign}k, \qquad (41)$$

$$\left(\frac{4\lambda^2\pi^2\rho_0^2}{(\lambda-1)^2} - k^2\right)\tilde{G}(k) = \frac{k^2}{\lambda-1} + \frac{2\pi\lambda\rho_0}{(\lambda-1)^2}|k|. \qquad (42)$$

We note that for $0 < \lambda < 1$, the correlation in the momentum space $\tilde{G}(k)$ is positive and depends only on the absolute value of k. It can be written in the form

$$\tilde{G}(k) = \frac{k^2}{2\omega(k)}, \qquad (43)$$

with the dispersion $\omega(k)$ given by

$$\omega(k) = \lambda\pi\rho_0|k| - \frac{\lambda-1}{2}k^2. \qquad (44)$$

Then we obtain the expression for the correlation function $G(x-y)$

$$G(x-y) = \frac{\rho_0}{2\pi(\lambda-1)}\int \mathrm{d}k \frac{e^{\mathrm{i}k(x-y)}|k|}{\frac{2\lambda}{\lambda-1}k_f - |k|}, \qquad (45)$$

with $k_f = \pi\rho_0$, representing the Fermi momentum. The integral can be recast as follows:

$$G(x-y) = -\frac{\rho_0}{\lambda-1}\delta(x-y) + \frac{2\lambda\rho_0^2}{(\lambda-1)^2}\int_0^\infty \mathrm{d}k \frac{\cos k(x-y)}{\frac{2\lambda}{\lambda-1}k_f - |k|}. \qquad (46)$$

Using the table of integrals [9], the correlation $G(x-y)$ turns out to be

$$G(x-y) = -\frac{\rho_0}{\lambda-1}\delta(x-y)+$$

$$+\frac{2\lambda\rho_0^2}{(\lambda-1)^2}\left[ci(\alpha(x-y))\cos(\alpha(x-y)) + si(\alpha(x-y))\sin(\alpha(x-y))\right], \tag{47}$$

where $\alpha = 2\lambda k_f/(\lambda-1)$ and $ci(x)$ and $si(x)$ denote the sine and cosine integral functions, respectively. In the fermionic case, $\lambda = 1$, the correlation function $G(x,y)$ given by relation (36) reduces to

$$G(x-y) = -\frac{1}{2\pi^2}\frac{1}{(x-y)^2}, \tag{48}$$

which in the momentum space agrees with the correlation function found by the authors of Ref. 10 in the $|k| < 2k_f$ sector. In our approach, the other sector $|k| > 2k_f$ is absent because of the large-N limit ($k_f = \pi\rho_0 \longrightarrow \infty$). In the $\lambda = 1/2(2)$ case, by expanding the correlation function (43) in the powers of $|k|/k_f$ up to the cubic terms, we easily obtain the result of Ref. 10, again in the $|k| < 2k_f(4k_f)$ sector only.

3.2. The eigenvalues and the eigenfunctions

Turning back to the Calogero model with confining interaction, let us show that there is a cancelation between the divergent vacuum energy $1/2\sum_n \omega_n$ of the harmonic Hamiltonian in Eq. (23) and the divergent last two terms. To this end, we compute the vacuum energy as

$$E_{vac} = \frac{\langle 0|H^{(2)}|0\rangle}{\langle 0|0\rangle}, \tag{49}$$

but this time with the vacuum $|0\rangle$ and the harmonic part $H^{(2)}$ of the Hamiltonian (23) given in terms of the density fluctuation $\eta(x)$. Using Eq. (7) for $\rho_0(x)$, we, finally, obtain

$$E_{vac} = \frac{\lambda-1}{4}\int \mathrm{d}x\partial_x^2\delta(x-y)|_{y=x} + \frac{\lambda}{2}\int \mathrm{d}x\rho_0(x)\partial_x\frac{P}{x-y}|_{y=x} - \frac{\omega}{2}\sqrt{\frac{N}{2}}. \tag{50}$$

Hence, the total Hamiltonian becomes

$$H = \frac{\omega}{2}\sqrt{\frac{N}{2}}[\lambda N(N-1) + N - 1] + \sum_n \omega_n a_n^\dagger a_n, \tag{51}$$

where we have introduced the standard creation and annihilation bosonic operators

$$a_n^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_n}q_n - \frac{\mathrm{i}}{\sqrt{\omega_n}}p_n\right), \tag{52a}$$

$$a_n = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_n}q_n + \frac{i}{\sqrt{\omega_n}}p_n\right). \tag{52b}$$

The ground-state energy obtained in Eq. (51) coincides with the Calogero results [1]. To find the eigenfrequencies ω_n, let us for the moment suppose that the solution $\varphi_n(x)$ of the eigenequation (25) can be cast into the form

$$\varphi_n(x) \sim \rho_0(x)x^n, \tag{53}$$

where n is a nonnegative integer. In this case we can evaluate the integral on the right-hand side of the eigenequation (25) using the formula

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}. \tag{54}$$

Introducing the moments m_k of the distribution function $\rho_0(x)$

$$m_k = \int \mathrm{d}x x^k \rho_0(x), \tag{55}$$

and using Eq. (7) for $\rho_0(x)$, we can reduce the right-hand side of Eq. (25) to

$$\omega\sqrt{\frac{N}{2}}(n+1)\rho_0(x)x^n + \rho_0(x)x^{n-2}\left[\frac{\lambda-1}{2}n(n-1) - \lambda(n-1)m_0\right] -$$

$$-\lambda\rho_0(x)x^{n-4}(n-3)m_2 + \cdots + \lambda\rho_0(x)m_{n-2} - \frac{\omega}{\sqrt{2N}}m_n\rho_0(x). \tag{56}$$

The structure of this expression indicates that the exact eigenfunction $\varphi_n(x)$ is given by an appropriate polynomal of x multiplied by the function $\rho_0(x)$:

$$\varphi_n(x) = \rho_0(x)\sum_{p=0}^{n} c_p^n x^p. \tag{57}$$

Substituting this expression into Eq. (25) and matching the coefficients of each power of x, we obtain an algebraic system of $n+1$ homogeneous equations in c_p^n:

$$c_n^n(n + 1 - \tilde{\omega}_n) = 0,$$

$$c_{n-1}^n(n - \tilde{\omega}_n) = 0,$$

$$c^n_{n-2}(n-1-\tilde{\omega}_n)+c^n_n\left(\frac{\lambda-1}{2}n(n-1)-\lambda m_0(n-1)\right)=0,$$

$$c^n_{n-3}(n-2-\tilde{\omega}_n)+c^n_{n-1}\left(\frac{\lambda-1}{2}(n-1)(n-2)-\lambda m_0(n-2)\right)=0,$$

$$c^n_{n-4}(n-3-\tilde{\omega}_n)+c^n_{n-2}\left(\frac{\lambda-1}{2}(n-2)(n-3)-\lambda m_0(n-3)\right)-\lambda c^n_n m_2(n-3)=0,$$

$$\vdots$$

$$c^n_2(3-\tilde{\omega}_n)+6(\lambda-1)c^n_4-3\lambda\sum_{k=0}^{n'}c^n_{2k+4}m_{2k}=0,$$

$$c^n_1(2-\tilde{\omega}_n)+3(\lambda-1)c^n_3-2\lambda\sum_{k=0}^{n'}c^n_{2k+3}m_{2k}=0,$$

$$c^n_0(1-\tilde{\omega}_n)+(\lambda-1)c^n_2-\lambda\sum_{k=0}^{n'}c^n_{2k+2}m_{2k}-\frac{1}{N}\sum_{k=0}^{n'+1}c^n_{2k}m_{2k}=0, \tag{58}$$

where $\omega_n=\tilde{\omega}_n\,\omega\sqrt{N/2}$, and n' is

$$n'=\begin{cases}\frac{n-2}{2} & \text{for } n \text{ even}\\ \frac{n-3}{2} & \text{for } n \text{ odd.}\end{cases}$$

The moments m_k vanish for k odd, because the distribution $\rho_0(x)$ is a symmetric function. A system of homogeneous equations has a nontrivial solution if its determinant vanishes. The determinant of our system (58) is just a product of diagonal terms (matrix of our system of equations is an upper-triangle matrix)

$$D=(\tilde{\omega}_n-n-1)(\tilde{\omega}_n-n)(\tilde{\omega}_n-n+1)\cdots(\tilde{\omega}_n-3)(\tilde{\omega}_n-2)\tilde{\omega}_n, \tag{59}$$

so it is easy to see that the eigenfrequencies ω_n are

$$\omega_n=\begin{cases}0 \quad \text{for } n=0\\ (n+1)\,\omega\sqrt{\frac{N}{2}} \quad \text{for } n\neq 0.\end{cases} \tag{60}$$

The excitation spectrum of the Calogero model is thus determined by the diagonalised Hamiltonian

$$H_{exc}=\sum_{n=1}^{\infty}\omega\sqrt{\frac{N}{2}}(n+1)a^\dagger_n a_n. \tag{61}$$

The general excited-state vector can be built by repeatedly acting with the creation operator (52a) on the vacuum state:

$$S\left\{\prod_{n}^{\infty}(a_n^\dagger)^{g_n}\right\}|0\rangle \sim S\left\{\prod_{n}^{\infty}H_{g_n}(\sqrt{\omega_n}q_n)\right\}|0\rangle. \tag{62}$$

The integer g_n represents the occupation numbers of the n–th oscillator mode, H_{g_n} is the Hermite polynomial of order g_n and S denotes total symmetrization. The corresponding excitation energy is

$$E_{\{g_n\}} = \omega\sqrt{\frac{N}{2}}\sum_{n=1}^{\infty}(n+1)g_n. \tag{63}$$

This is again in complete accordance with the Calogero result, the only difference being in $N \to \infty$. It is interesting to note that the excitation spectrum (63) does not depend on the statistical parameter λ.

Let us now list the first few eigenfuctions $\varphi_n(x)$ and the corresponding eigenfrequencies:

$$\begin{gathered}
\varphi_0(x) = c_0^0\rho_0(x), \quad \omega_0 = 0, \\
\varphi_1(x) = c_1^1 x\rho_0(x), \quad \omega_1 = 2\omega\sqrt{\frac{N}{2}}, \\
\varphi_2(x) = (c_0^2 + c_2^2x^2)\rho_0(x), \quad \omega_2 = 3\omega\sqrt{\frac{N}{2}}, \\
\varphi_3(x) = (c_1^3x + c_3^3x^3)\rho_0(x), \quad \omega_3 = 4\omega\sqrt{\frac{N}{2}}, \\
\varphi_4(x) = (c_0^4 + c_2^4x^2 + c_4^4x^4)\rho_0(x), \quad \omega_4 = 5\omega\sqrt{\frac{N}{2}}, \\
\varphi_5(x) = (c_1^5x + c_3^5x^3 + c_5^5x^5)\rho_0(x), \quad \omega_5 = 6\omega\sqrt{\frac{N}{2}}, \\
\vdots
\end{gathered} \tag{64}$$

The fact that Eq. (25) has an eigensolution $\varphi_0(x)$ with a vanishing eigenvalue ω_0 is a consequence of the translational invariance of the Calogero model. We note that the polynomial structures in $\varphi_n(x)$ have only even powers of x or only odd powers of x, depending on n being even or odd. The coefficients c_p^n are interrelated by the system (58) and the free ones can be determined from the normalization condition (22a). For example,

$$3\omega\sqrt{\frac{N}{2}}c_0^2 = c_2^2(-\lambda N + \lambda - 1 - \omega\frac{m_2}{\sqrt{2N}}),$$

$$2\omega\sqrt{\frac{N}{2}}c_1^3 = c_3^3(-2\lambda N + 3(\lambda - 1)),$$

$$2\omega\sqrt{\frac{N}{2}}c_2^4 = c_4^4(-3\lambda N + 6(\lambda - 1)),$$

$$5\omega\sqrt{\frac{N}{2}}c_0^4 = c_4^4(-\lambda m_2 - \omega\frac{m_4}{\sqrt{2N}}) + c_2^4(-\lambda N + \lambda - 1 - \omega\frac{m_2}{\sqrt{2N}}),$$

$$2\omega\sqrt{\frac{N}{2}}c_3^5 = c_5^5(-4\lambda N + 10(\lambda - 1)),$$

$$4\omega\sqrt{\frac{N}{2}}c_1^5 = c_3^5(-2\lambda N + 3(\lambda - 1)) - c_5^5 2\lambda m_2,$$

$$\vdots \tag{65}$$

All moments m_k can be evaluated using the recurrence relation

$$m_k = \frac{1}{\omega}\sqrt{\frac{2}{N}}\left\{\left[-(\lambda - 1)\frac{k-1}{2}\right]m_{k-2} + \lambda\sum_{i=1}^{k/2-1} m_{k-2i}m_{2i-2} - \frac{\lambda}{2}m_{k/2-1}^2\right\}, \tag{66}$$

which follows from Eq. (7) and formula (54). Here we list few moments:

$$m_0 = N,$$

$$m_2 = \frac{1}{\omega}\sqrt{\frac{N}{2}}(\lambda N - \lambda + 1),$$

$$m_4 = \frac{1}{\omega^2}\left[\lambda N - \frac{3}{2}(\lambda - 1)\right](\lambda N - \lambda + 1). \tag{67}$$

We can go on with this procedure. It is obvious that a systematic construction of all polynomial solutions to the eigenequation (25) can be accomplished.

4. Comments

Having discussed the eigenfunctions and the corresponding eigenvalues of Eq. (25) for the general value of the statistical parameter λ, we should briefly analyse the boson (λ=0) and the fermion (λ=1) case separately. For λ=0, using the explicit form for ρ_0 [Eq. (11)], it is easily verified that the eigenequation (25) reduces to the differential equation for the Hermite polynomials. So φ_n is given by

$$\varphi_n(x) = \rho_0(x) H_n\left(\sqrt{\omega\sqrt{\frac{N}{2}}}\, x\right), \quad n \geq 1, \tag{68}$$

which can be easily recovered from the set (65). This is in accordance with the result obtained by Jevicki and Sakita in Ref. 11, for a large number of harmonic oscillators. For the fermionic case (or the matrix model), $\lambda = 1$, the equation effectively reduces to the well-known formula for the Hilbert transform of the Chebyshev polynomials U_n and T_n

$$\fint_{-1}^{+1} \frac{T_n(y)\mathrm{d}y}{(y-x)\sqrt{1-y^2}} = \pi U_{n-1}(x). \tag{69}$$

Using the explicit form for ρ_0 [Eq. (12)], one can show that φ_n reduces to

$$\varphi_n(x) = \sin\left(\frac{n\pi}{T} t(x)\right) = \rho_0(x) U_n\left(\sqrt{\frac{\omega}{2}\sqrt{\frac{2}{N}}}\, x\right), \quad n \geq 1, \tag{70}$$

where $t(x)$ is the time of flight of a classical particle in the harmonic potential, with T being the semiperiod. This is in exact agreement with the calculations given in Ref. 12. It can be verified that our set (65) correctly reproduces the coefficients for the Chebyshev polynomials U_n. Hence, our set of orthogonal polynomials correctly interpolates between the two well-known cases of bosons and fermions and, for intermediate values of λ, gives the excitation spectrum and the corresponding normal modes for particles with fractional statistics.

It is interesting that the fermionic case ($\lambda = 1$) is closely related to the random-matrix theory which does not possess kinetic term $\partial_x \rho/\rho$ in the collective Hamiltonian (1). This theory describes dynamics of the eigenvalues of the orthogonal, unitary and symplectic matrices. Recently, the authors of Ref. 13 reported a simple result for the correlation of the eigenvalue density at two points in the spectrum. Now, we are going to show that our correlation function $G(x,y)$, calculated up to the quadratic term in the fluctuations around the semiclassical configuration $\rho_0(x)$, is in agreement with the Brézin-Zee spectral correlator given in Ref. 13. Having in mind the relations for $G(x,y)$ (34), $\varphi_n(z)$ (70), and ω_n (60), we easily obtain

$$G(x,y) = \frac{T}{4\pi^2} \frac{1}{\rho_0(x)\rho_0(y)} \partial_t \partial_{t'} \left(\sum_n \frac{\cos\frac{n\pi}{T}(t-t')}{n} - \sum_n \frac{\cos\frac{n\pi}{T}(t+t')}{n} \right). \tag{71}$$

Next, we employ the summation formula [9]

$$\sum_n \frac{\cos n\alpha}{n} = -\frac{1}{2}\ln 2(1-\cos\alpha), \tag{72}$$

so that $G(x,y)$ can be rewritten as

$$G(x,y) = -\frac{1}{4T\rho_0(x)\rho_0(y)}\,\frac{1-\cos\frac{\pi}{T}t(x)\cos\frac{\pi}{T}t(y)}{\left(\cos\frac{\pi}{T}t(x)-\cos\frac{\pi}{T}t(y)\right)^2}. \tag{73}$$

In our case (12) the end points of the spectrum are given by $-a$ and a, and the time of flight than reads

$$t(x) = \int\limits_{-a}^{x}\frac{\mathrm{d}y}{\rho_0(y)} = \int\limits_{-a}^{x}\frac{\mathrm{d}y}{\frac{\pi}{T}\sqrt{a^2-y^2}} = \frac{T}{\pi}\left(\arcsin\frac{x}{a}+\frac{\pi}{2}\right). \tag{74}$$

Inverting this relation, we obtain

$$x = -a\cos\frac{\pi}{T}t(x). \tag{75}$$

Combination of Eqs. (73) and (75) yields the exact two-point correlation function of Ref. 13

$$G(x,y) = -\frac{1}{4T\rho_0(x)\rho_0(y)}\,\frac{a^2-xy}{(x-y)^2}, \tag{76}$$

valid for any even potential in which the eigenvalues ”move”. We note in passing that this result can be rederived from Eq. (36) simply by putting $\lambda = 1$.

Finally, it would be interesting to expand the collective Hamiltonian (1) up to the cubic terms in the fluctuations around the semiclassical configuration $\rho_0(x)$. These cubic interaction terms would provide a basis for systematic perturbative computations of scattering amplitudes, higher (loop) corrections to the ground-state energy and the dispersion law.

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POBUĐENJA KOLEKTIVNE TEORIJE POLJA U CALOGEROVU MODELU

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Razmatran je veliko-N Calogerov model u pristupu kolektivne teorije polja, zasnovanom na $1/N$ razvoju. Bogomolnyev limes daje jednadžbu za poluklasičnu konfiguraciju, kao i točnu energiju osnovnog stanja. Korištenjem metode ortogonalnih polinoma, nađen je spektar pobuđenja malih fluktuacija gustoće oko poluklasičnog rješenja, za bilo koju vrijednost statističkog parametra λ. Eksplicitno je izvedena valna funkcija pobuđenih stanja kao umnožak Hermiteovih polinoma. Dvotočkasta korelaciona funkcija je izračunata kao $1/\rho$ razvoj, za bilo koju intermedijarnu statistiku.