ISSN 1330–0016
CODEN FIZBE7

CANONICAL AND NON CANONICAL QUANTUM PHASE

MATTEO G. A. PARIS

Dipartimento di Fisica "Alessandro Volta" dell'Universitá di Pavia, via Bassi 6, I-27100 Pavia, ITALIA, E-mail: PARIS@PV.INFN.IT and Arbeitsgruppe 'Nichtklassische Strahlung' der Max-Planck-Gesellschaft, Rudower Chaussee 5, D-12489 Berlin, Germany

Received 7 January 1997

Revised manuscript received 28 March 1997

UDC 530.145

PACS 03.65.Bz

The so-called canonical phase variable for the quantized harmonic oscillator is usually derived from quantum estimation theory or by a limiting procedure in the Fock space. In this paper we show that, using the full content of Born's statistical rule, it can be uniquely derived from correspondence principle, in the framework of elementary quantum mechanics. Difficulties of the Pegg-Barnett approach are illustrated and *non canonical* phase variables arising from Mandel's experiments are also discussed.

1. Introduction

The quantum description of the phase of the harmonic oscillator has been debated for a long time and many different approaches have been suggested [1]. The problem has a fundamental interest but is also relevant for applications. This is mainly due to the fact that a single-mode radiation field can be usefully modeled as a unit-mass harmonic oscillator. As a matter of fact, no selfadjoint phase operator can be defined through Poisson bracket quantization [2], and this has provoked the diffuse conviction that phase does not correspond to a proper quantum variable [3]. Nevertheless, there is a general agreement that the probability

$$\mathrm{d}P(\varphi) = \frac{\mathrm{d}\varphi}{2\pi} \sum_{n,m=0}^{\infty} \rho_{nm} \exp\{\mathrm{i}(n-m)\varphi\} , \tag{1}$$

plays a special role among the different proposals. In fact, it is often referred to as *canonical* phase distribution [4] for the quantum state described by the density operator $\hat{\rho}$. Distribution in Eq. (1) was earlier introduced by London [5]. It can be viewed as the limit of the distribution in Pegg-Barnett approach [6], which is related to a truncation in the dimension of the Hilbert space of the harmonic oscillator. This approach allows evaluation of the expectation values after a limiting procedure in the dimensional variable. However, it cannot be used for defining a proper phase variable in a strict sense. The Fock space of the harmonic oscillator is basically infinite dimensional and any other assumption would require a different definition for the photon [7]. Distribution (1) has been also derived by Helstrom [8] and Holevo [9] in the more general framework of quantum estimation theory. There the phase is considered as an external parameter, which has to be inferred from the measurement of some phase-related observable. This approach has been fruitful in deriving lower bounds on the precision of ideal and feasible phase detection schemes [10]. However, definition of canonical quantum phase is beyond its scope. In fact, for setting the proper estimation framework, it introduces further external *a priori* assumptions not related with the quantum mechanics.

On the other hand, it is a purpose of this paper to show that using the full statistical content of Born's statistical rule, the proper quantum phase of Eq. (1) can be uniquely defined, from the correspondence principle, in the framework of elementary quantum mechanics.

The main part of the paper is Section 2, where the canonical quantum phase variable is derived starting from the correspondence principle. From this result some basic facts about the quantum phase are derived and presented in section 3. In Section 4, the Pegg-Barnett approach is briefly reviewed, in order to show it should be considered as an useful mathematical trick to derive phase statistics, while it it cannot be used for defining a proper phase variable in a strict sense. In Section 5, the *non canonical* phase variables arising from Mandel's experiments are analysed.

2. Canonical quantum phase

The whole probabilistic structure of quantum mechanics is contained in the Born's statistical rule

$$\mathrm{d}P[\hat{\rho}](z) = \mathrm{Tr}\{\hat{\rho}\ \mathrm{d}\hat{\mu}(z)\} , \tag{2}$$

which assures propagation of convex linear combinations from density operators toward probabilities. The Born's rule defines a genuine probability density distribution $\mathrm{d}P[\hat{\rho}](z)$ if $\mathrm{d}\hat{\mu}(z)$ satisfies the axioms for a *probability operator measure* (POM), namely, that it is nonegative (hence selfadjoint), $\mathrm{d}\hat{\mu}(z) \geq 0$, and provides a resolution of identity on the set of possible outcomes

$$\int_{\mathcal{Z}} d\hat{\mu}(z) = \hat{\mathbf{1}} \,. \tag{3}$$

Equation (3) guarantees probability density (2) to be normalized. The spectral resolution $d\hat{E}(z) = |z\rangle\langle z|\, dz$ of a selfadjoint operator $\hat{Z}$ provides a *projection valued measure* (PVM) which belongs to the class of POMs. The converse is not true, namely POMs constitute a more general class of operators than PVMs.

It is a matter of fact that quantum prescriptions to obtain probability distributions are satisfied by mathematical objects (the POMs) which are more general than PVMs obtained from selfadjoint operators. However, a POM is necessarily (Naimark Theorem) [11] a partial trace of a PVM coming from a selfadjoint operator defined on a larger Hilbert space. This provides the link with the other postulates of quantum mechanics, especially with the dictum “Only observables corresponding to selfadjoint operators can be measured”. A quantum measurement process, in fact, is an interaction among the system under examination and a set of measuring systems unitarily referred to as the *probe* of the measurement device. According to the above stated postulate, we can measure a variable when it is associated to a selfadjoint operator $\hat{Y}$ defined on the Hilbert $\mathcal{H}' = \mathcal{H}_P \otimes \mathcal{H}_S$ space, describing the ”system plus probe” quantum states. By tracing the spectral measure of $\hat{Y}$ over the probe degree of freedom, we obtain the POM of the measurements

$$d\hat{\mu}(y) = dy \mathrm{Tr}_P \left\{ |y\rangle\langle y|\, \hat{1}_S \otimes \hat{\rho}_P \right\} \,, \tag{4}$$

which represents the operatorial description of the measurement itself restricted to the Hilbert space $\mathcal{H}_S$ of the system. After this partial trace, the resulting probability measure could be still a PVM. In general it is a POM. When it is a PVM, as it happens for observables as position, momentum and energy, we can speak about observables without referring to a particular measurement apparatus.

However, this is not the case of a quantum phase that unavoidably is described by a POM which is not reducible to a PVM. Such a situation is often summarized by two assertions: (i) different measuring systems measure different phase variables [10,12] and (ii) there is not a ”canonical phase variable in quantum mechanics”. The first assertion is a general feature of quantum mechanics and it is certainly correct, whereas we will show the second one to be incorrect. The requirement of a phase measurement to be described by a POM will be sufficient to derive a unique phase variable (Heisenberg conjugate to the number operator) from the correspondence principle.

The quantum mechanical description of the harmonic oscillator is based on annihilation, a, and creation, $a^\dagger$, operators which form the number operator $\hat{N} = a^\dagger a$. Multiple application of $a^\dagger$ to the vacuum state leads to the Fock basis $|n\rangle = (n!)^{-1/2} a^{\dagger n}|0\rangle$ which span the whole Hilbert space. Number states represent the eigenstates of the number operator, whose spectrum coincides with the set of natural numbers $\mathcal{N} = 0, 1, \ldots$.

The classical complex amplitude α of harmonic oscillations can be decomposed using the polar variable $\alpha = |\alpha| \exp\{i\varphi\}$, $|\alpha|^2$ being the intensity of oscillations and φ their phase. The correspondence principle suggests to try a similar decomposition on the annihilation operator

$$a = \hat{E}\sqrt{\hat{N}}\,, \tag{5}$$

$\hat{E}$ being the quantum phase factor. From Eq. (5) it is not possible to extract a phase observable. A direct inspection, in fact, shows that

$$\hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{6}$$

$$\hat{E}\hat{E}^\dagger = \hat{\mathbf{1}} \qquad \hat{E}^\dagger\hat{E} = \hat{\mathbf{1}} - |0\rangle\langle 0|\,, \tag{7}$$

namely that $\hat{E}$ is not unitary. Therefore, the formula $\hat{E} = \exp\{i\hat{\varphi}\}$ cannot define any self-adjoint operator. In other words, the operatorial equation

$$\hat{E} = \int_{-\pi}^{\pi} \exp\{i\varphi\}\, d\hat{E}(\varphi) \tag{8}$$

is not satisfied by any PVM $d\hat{E}(\varphi)$. We now look for a solution of the operatorial equation

$$\hat{E} = \int_{-\pi}^{\pi} \exp\{i\varphi\}\, d\hat{\mu}(\varphi)\,, \tag{9}$$

describing the phase factor in terms of a phase POM, $d\hat{\mu}(\varphi)$.

We first note that for any state vector $|\psi\rangle$

$$||\hat{E}\psi|| < ||\psi||\,, \tag{10}$$

where $||\psi|| = \sqrt{\langle\psi|\psi\rangle}$ denotes the norm of a vector in the Hilbert space. Eq. (10) means that $\hat{E}$ belongs to the class of contraction. For such a kind of transformation maps in the Hilbert space, a "quasi-spectral" theorem is known assuring that [13–15]

$$\hat{E}^n = \int_{-\pi}^{\pi} \exp\{in\varphi\}\, d\hat{\mu}(\varphi)\,, \tag{11}$$

possesses a unique solution. This theorem, together with POM's description of a phase measurement, demonstrates the existence of a unique canonical phase variable coming from the correspondence principle. In order to derive its explicit expression, we observe that the eigenstates of $\hat{E}$ are of the form

$$|\zeta\rangle = \sum_{n=0}^{\infty} \zeta^n |n\rangle \zeta = |\zeta| \exp\{i\varphi\} \qquad 0 \le |\zeta| \le 1\,. \tag{12}$$

It is simple to check that only eigenstates with $|\zeta| = 1$ can satisfy (11) so that the canonical phase POM is given by

$$\begin{aligned} \mathrm{d}\hat{\mu}(\varphi) &= \frac{\mathrm{d}\varphi}{2\pi}|e^{\mathrm{i}\varphi}\rangle\langle e^{\mathrm{i}\varphi}| = \\ &= \frac{\mathrm{d}\varphi}{2\pi}\sum_{n,m=0}^{\infty}\exp\{\mathrm{i}(n-m)\varphi\}|n\rangle\langle m| \, . \end{aligned} \tag{13}$$

Thus, a quantum state $\hat{\rho}$ is characterized by the canonical phase probability Eq. (1). To prove this statement, we used only Born's rule and the correspondence principle. This is the main result of this paper.

3. *Some facts about the quantum phase*

As prescripted by Naimark's theorem, canonical POM (13) has to be the partial trace of some PVM in a larger Hilbert space than that of the harmonic oscillator. Two explicit examples of such an extension have been given in Refs. 16 and 17.

A phase operator can be defined as [18]

$$\begin{aligned} \hat{\Phi} &= \int_{-\pi}^{\pi}\varphi\, \mathrm{d}\hat{\mu}(\varphi) = \\ &= -\mathrm{i}\sum_{n\neq m}(-)^{n-m}\frac{1}{n-m}|n\rangle\langle m| \, . \end{aligned} \tag{14}$$

The Dirac number-phase bracket $[\hat{\Phi},\hat{N}]|\psi\rangle = \mathrm{i}|\psi\rangle$ is defined for $|\psi\rangle \in \mathcal{D}$

$$\mathcal{D} = \{|\psi\rangle \in \mathcal{D} \mid \sum_{n=0}^{\infty}\langle n|\psi\rangle = 0\ \} \ , \tag{15}$$

which excludes the number states. The domain $\mathcal{D}$ is dense in the Hilbert space, thus it is enough for a good definition of the Heisenberg commutation relation. The exclusion of the number states is necessary to have well defined uncertainty relations. Number states, in fact, have zero variance in the number itself, whereas phase variance has to be finite for topological reasons. Canonical phase POM is shift-covariant, namely

$$\mathrm{e}^{\mathrm{i}\widehat{N}\varphi_0}\, \mathrm{d}\hat{\mu}(\varphi)\, \mathrm{e}^{-\mathrm{i}\widehat{N}\varphi_0} = \mathrm{d}\hat{\mu}\,[(\varphi_0+\varphi)_{\mathrm{mod}\,2\pi}] \ ,$$

however, it is not idempotent (as it happens for a PVM) $[\mathrm{d}\hat{\mu}(\varphi),\mathrm{d}\hat{\mu}(\varphi')]\neq 0$ and this leads to failure in the function calculus [19]

$$\widehat{F(\varphi)} \equiv \int_{-\pi}^{\pi} f(\varphi)\, \mathrm{d}\hat{\mu}(\varphi) \neq f(\widehat{\Phi})$$

$$\equiv \quad f\left(\int_{-\pi}^{\pi} \varphi \, \mathrm{d}\hat{\mu}(\varphi)\right) . \tag{16}$$

The actual phase statistics is provided by POM definition of the operator function, as shown by the following identities

$$\begin{aligned} \langle f(\varphi)\rangle &= \int_{-\pi}^{\pi} \mathrm{d}P(\varphi) f(\varphi) = \\ &= \int_{-\pi}^{\pi} \mathrm{Tr}\{\hat{\rho}\mathrm{d}\hat{\mu}(\varphi)\} f(\varphi) = \\ &= \mathrm{Tr}\left\{\hat{\rho}\int_{-\pi}^{\pi} f(\varphi)\mathrm{d}\hat{\mu}(\varphi)\right\} = \\ &= \mathrm{Tr}\left\{\hat{\rho}\widehat{F(\varphi)}\right\} . \end{aligned} \tag{17}$$

4. About the Pegg and Barnett approach

In the Pegg-Barnett approach [6], the use of the truncated expansion is suggested

$$|\phi_m\rangle_s = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp \mathrm{i}\phi_m |n\rangle ,$$
$$\phi_m = \phi_0 + \frac{2m\pi}{s+1} , \quad m = 0, 1, \ldots, s+1; , \tag{18}$$

as phase states providing the discrete PVM

$$\hat{M}_s(\phi_m) = |\phi_m\rangle\langle\phi_m| \, \frac{2m\pi}{s+1} , \tag{19}$$

in the $s+1$-dimensional Hilbert space $\mathcal{H}_s$ spanned by $|n\rangle$, $n = 0, 1, \ldots, s+1$. In this way they can define a selfadjoint operator in $\mathcal{H}_s$:

$$\hat{\phi}_s = \sum_{m=0}^{s} \phi_m \, |\phi_m\rangle\langle\phi_m| , \tag{20}$$

and in the number expansion

$$\hat{\phi}_s = \phi_0 + \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{n\neq m}^{s} \frac{e^{\mathrm{i}(n-m)\phi_0}}{\exp\left(\frac{2\pi \mathrm{i}(n-m)}{s+1}\right) - 1} |n\rangle\langle m| . \tag{21}$$

Unfortunately, $\hat{M}_s(\phi_m)$ does not converge neither to a PVM nor to a POM in $\mathcal{H}_\infty$

$$\lim_{s\to\infty} \hat{M}_s(\phi_m) \begin{array}{l} \neq d\hat{E}(\varphi) \\ \neq d\hat{\mu}(\varphi) \end{array} , \tag{22}$$

thus one can not define any phase variable in the whole infinite-dimensional Hilbert space of the harmonic oscillator.

However, a weaker form of convergence should be noticed. Any average of operator phase function

$$\widehat{f(\phi_s)} = \sum_{m=0}^{s} f(\phi_m)\,|\phi_m\rangle\langle\phi_m| , \tag{23}$$

converges, in fact, to the canonical one defined in Eqs. (16) and (17)

$$\begin{aligned} \langle \hat{f}_s(\varphi)\rangle &= \mathrm{Tr}\left\{\hat{\rho}\sum_{m=0}^{s} f(\phi_m)|\phi_m\rangle\langle\phi_m|\right\} \xrightarrow{s\to\infty} \\ \langle \widehat{f}(\varphi)\rangle &= \mathrm{Tr}\left\{\hat{\rho}\int_{-\pi}^{\pi} f(\varphi)\mathrm{d}\hat{\mu}(\varphi)\right\} , \end{aligned} \tag{24}$$

where $\mathrm{d}\hat{\mu}(\varphi)$ is the canonical POM. The set of Eqs. (18)-(23) together with the prescription of Eq. (24) represent the Pegg-Barnett description of the quantum phase. However, the phase operator in Eq. (20) is neither selfadjoint nor it allows for a selfadjoint extension in the infinite dimensional Fock space of the harmonic oscillator [1]. Pegg and Barnett ask only for a selfadjoint operator in the *finite-dimensional* Hilbert space, whereas the physical expectation values have to be evaluated with the *ad hoc* prescription given by Eq. (24). They are actually using a novel definition for the quantum electrodynamics ! [7]. In QED, in fact, the photonic field has to be described in an infinite-dimensional space and any other description cannot correspond to a photon. For these reasons I does not consider this approach of fundamental interest. Its convenience as a useful mathematical trick is, however, justifyied by the weak convergence expressed in Eq. (24).

[1]This problem reminds of one defininining the canonical momentum of a particle. The operator $\hat{p} = -\mathrm{i}\partial_x$, in fact, is selfadjoint over $\mathcal{L}^2(\mathbf{R})$, whereas it allows for a selfadjoint extension over $\mathcal{L}^2([a,b])$. On the contrary, it is neither selfadjoint nor it allows for a selfadjoint extension over $\mathcal{L}^2(\mathbf{R}^+)$.

5. *Non-canonical phase variables*

A set of interesting experiments about the phase measurements on coherent states have been performed by Mandel's group in Rochester [20]. They use an eight-port homodyne detection scheme, which provides a way for simultaneously measuring a couple of photocurrents, and consider different preparations for the probe modes of the apparatus. This results in different determinations of the phase, which they consider as an evidence for a non-unique definition of the quantum optical phase. A comment is, however, in order: while the fact that different measuring systems yield different phase-related quantities appears rather obvious, this does not mean that it is not possible to uniquely define the canonical variable corresponding to the phase. We will return on this point later.

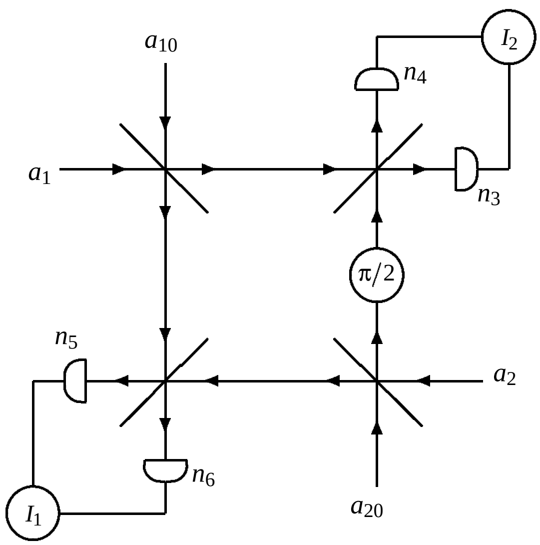

Fig. 1. Eight-port homodyne detection for non-canonical phase measurements.

The schematic diagram of the Mandel's detector is shown in Fig. 1. There are four 50photocounters, whereas a $\pi/2$ phase shifter is inserted in one arm. The signal mode is a, whereas a_{10} and a_{20} are unexcited modes, namely they are placed in the vacuum. The mode a_2 can be excited in both a weak and a very strong coherent state. Thus, we have to keep in mind that we are actually dealing with two different experimental systems.

Each experimental event consists of a simultaneous detection of two commuting difference-photocurrents

$$\hat{I}_1 = \frac{\hat{n}_4 - \hat{n}_3}{\sqrt{(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2}}$$
$$\hat{I}_2 = \frac{\hat{n}_6 - \hat{n}_5}{\sqrt{(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2}} , \qquad (25)$$

which are referred to as *cosine* and *sine* operator.
For the mode a_2, excited in strong coherent state, it is possible to derive the explicit phase POM of the detector [10]. In this case, in fact, the reduced photocurrents

$$\hat{I}_1 = \frac{\hat{I}_1}{|z|} , \qquad \hat{I}_2 = \frac{\hat{I}_2}{|z|} \tag{26}$$

are continuos variables which trace two conjugated quadratures of the field. Each experimental outcome is a complex number $\alpha = I_2 + iI_1$ representing a point in the complex plane of the field amplitude. Thus, the measured phase can be defined as the polar angle of the point itself

$$\phi = \arctan\left[\frac{I_2}{I_1}\right] . \tag{27}$$

The apparatus density matrix is given by

$$\hat{\rho}_A = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |z\rangle\langle z| \ , \tag{28}$$

whereas the POM of the detector is evaluated as

$$d\hat{\mu}(\alpha,\overline{\alpha}) = \frac{d^2\alpha}{\pi}\int_{\mathbf{R}}\frac{d\mu}{2\pi}\int_{\mathbf{R}}\frac{d\nu}{2\pi}\,\mathrm{Tr}_A\left[\hat{\rho}_A \otimes \hat{\mathbf{1}}_S\, e^{\mathrm{i}\mu(I_1-\mathrm{Re}\alpha)+\mathrm{i}\nu(I_2-\mathrm{Im}\alpha)}\right] . \tag{29}$$

After some operator algebra, we arrive at the final result [10]

$$d\hat{\mu}(\alpha,\overline{\alpha}) = \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \ , \tag{30}$$

which means that the eight-port homodyne detector provides the generalized measurement of Husimi Q-function $Q(\alpha,\overline{\alpha}) = \langle\alpha|\hat{\rho}|\alpha\rangle$. The phase POM is the marginal POM of (30) integrated over the radius

$$d\mu(\phi) = \frac{d\phi}{2\pi}\sum_{n,m=0}^{\infty} e^{\mathrm{i}(n-m)\phi}\,\frac{\Gamma\left(\frac{1}{2}(n+m)+1\right)}{\sqrt{n!m!}}\rho_{nm} \ , \tag{31}$$

corresponding to the marginal phase probability given by

$$dP(\phi) = \frac{d\phi}{2\pi}\int_0^{\infty} d\rho\ \rho\ Q(\rho e^{\mathrm{i}\phi},\rho e^{-\mathrm{i}\phi}) \ . \tag{32}$$

The coefficients

$$\frac{\Gamma\left(\frac{1}{2}(n+m)+1\right)}{\sqrt{n!m!}}$$

are always lower than one, thus leading to a noisy phase determination, namely to a phase distribution which is broader than the canonical one for any quantum state of radiation [10]. This noise comes from the fundamental principle, as the eight-port apparatus measures two photocurrents which, in turn, trace two conjugated quadratures of the field, namely two non-commuting observables [21]. We conclude that the Mandel's detector leads to a *non canonical* phase measurement. This argument, however, cannot be used to claim non uniqueness of canonical phase variable. Let us clarify this point by a simple example. The canonical measurement of the field intensity is represented by the detection of the photon number POM $\hat{\mu}(n) = |n\rangle\langle n|$, being $|n\rangle$ number eigenstates. In realistic photodetectors, however, only a fraction of the incoming photons are revealed. Thus, what is really measured in a laboratory is the Bernoulli convoluted POM

$$\hat{\mu}(m)_\eta = \sum_{n=m}^{\infty} |n\rangle\langle n| \begin{pmatrix} n \\ m \end{pmatrix} \eta^m (1-\eta)^{n-m} \,, \tag{33}$$

η being the detection efficiency. The *non canonical* POM in Eq. (33) is very different from the canonical one, but nobody would raise the objection that $\hat{\mu}(n) = |n\rangle\langle n|$ does not represent the canonical quantum variable connected to the field intensity.

The case of the mode a_2, excited in a weak coherent state, belongs to a different class of problems. Actually, both a_1 and a_2 are fluctuating quantum fields, thus the measured phase is a two-mode phase-difference operator. The POM for such a measurement has not been evaluated yet. In any case, it could not be directly compared with the canonical POM presented in this paper, as the latter represents canonical phase variable in the single-mode case. Work along this line is in progress and the results will be presented elsewhere.

6. Conclusions

In conclusion, the POM in Eq. (13) has been shown to be canonical, namely it can be deduced from the correspondence principle without any assumption that quantum mechanics postulates. It has also been shown how this approach can comprise all the features ascribed to a proper phase variable. The Pegg-Barnett approach has been shown to be equivalent to the present canonical approach at the level of expectation values, whereas its difficulties at the fundamental level have been discussed in some details. Mandel's experiment with highly excited probe field has been shown to provide a noisy *non canonical* phase measurement. The weak-probe case represents the measurement of two-mode phase-difference operators and cannot be directly compared to the present approach, which provides the canonical POM for a single-mode phase variable.

Acknowledgement

I would thank Harry Paul for his kind hospitality in the group 'Nichtklassiche Strahlung' of the Max-Planck-Society. I would also thank Mladen Pavičić for inviting me in Zagreb and Ole Steuernagel for interesting discussions. This work has been partially supported by the University of Milan by a scholarship for postgraduate studies in foreign countries. Finally, I would like to thank Valentina De Renzi.

References

1) R. Lynch, Phys. Reports **256** (1995) 367;

2) L. Susskind and G. Glogower, Physics (New York) **1** (1964) 49;

3) M. G. A. Paris, Nuovo Cimento **B 111** (1996) 1151;

4) U. Leonhardt, J. A. Vaccaro, B. Böhmer and H. Paul, Phys. Rev. **A 51** (1995) 84;

5) F. London, Z. Phys. **40** (1927) 193, Eq. (13);

6) D. T. Pegg and S. M. Barnett, Europhys. Lett. **6** (1988) 483; Phys. Rev. **A 39** (1989) 1665; J. Mod. Optics **36** (1989) 7;

7) M. M. Nieto, Phys. Scr. **T48** (1993) 5;

8) C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York), 1976;

9) A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland Publishing, Amsterdam), 1982;

10) G. M. D'Ariano and M. G. A. Paris, Phys. Rev. **A 49** (19940 3022;

11) M. A. Naimark, Izv. Akad. Nauk SSSR Ser.Mat. **4** (1940) 227. See also Ref. 15;

12) P. Busch, M. Grabowski and P. J. Lahti, Lect. Notes, Phys. **31**, Springer, Berlin, 1995;

13) B. Sz-Nagy, Acta Sci. Math. (Szeged) **15** (1953) 87;

14) M. Foias, Bull. Soc. Math. (France) **85** (1957) 263;

15) W. Mlak, *Hilbert Spaces and Operator Theory,* Kluwer Academic, Dordrecht, 1991, p.235;

16) M. Ban, J. Opt. Soc. Am. **B 9** (1992) 1189;

17) J. H. Shapiro, in *The Workshop on Squeezed States and Uncertainty Relations*, NASA Conf. Public. 3135, Washington DC, 1992, p.107;

18) V. P. Popov and V. S. Yarunin, J. Mod. Opt. **39** (1992) 1525;

19) G. M. D'Ariano and M. G. A. Paris, Phys. Rev. **A 48** (1993) R4039;

20) J. W. Noh, A. Fougéres and L. Mandel, Phys. Rev. **A 45** (1992) 424, **A 46** (1993) 2480; J. R. Torgeson and L. Mandel, Phys. Rev. Lett. **76** (1996) 3939;

21) H. P. Yuen, 1982 Phys. Lett. **A 91** (1982) 101.

KANONSKA I NEKANONSKA KVANTNA FAZA

Varijabla kanonske faze za kvantiziran harmonički oscilator obično se izvodi na osnovi kvantne teorije ocjena ili graničnim postupkom u Fockovom prostoru. U ovom se radu pokazuje da se potpunom primjenom Bornovog statističkog pravila kvantna faza može jednoznačno izvesti iz načela korespondencije, u okviru elementarne kvantne mehanike. Razmatraju se teškoće Pegg-Barnettovog pristupa i raspravljaju nekanonske fazne varijable koje se izvode iz Mandelovih eksperimenata.